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COMMENTARIES ON THE PAPER *SOLENOIDAL MANIFOLDS* BY DENNIS SULLIVAN

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Abstract. Several remarks and comments on the paper about solenoidal manifolds referred in the title are given. In particular, the fact is emphasized that there is a parallel theory of compact solenoidal manifolds of dimensions one, two and three with the theory of compact manifolds of these dimensions.

A k-dimensional *solenoidal manifold* or *lamination* is a metric space which is locally the product of an euclidean k-disk and an infinite perfect and totally disconnected set (a subset of the Cantor set). These solenoidal manifolds appear naturally in many branches of mathematics. In topology the Vietoris-Van Dantzig solenoid ([13] [15]) is one of the fundamental examples in topology and it motivated the development of homology and cohomology theories which could apply to these spaces, for instance in the paper by Steenrod [10].

Solenoids appear naturally also as Pontryagin duals of discrete locally compact Hausdorff abelian groups. For instance if $\mathbb{Q}$ denotes the rationals with addition as group structure and with the discrete topology then its Pontryagin dual $\mathbb{Q}^*$ is the universal 1-dimensional solenoid which is a compact abelian group which fibers over the circle $\mathbb{S}^1$ via an epimorphism $p : \mathbb{Q}^* \to \mathbb{S}^1$ where the fibre is the Cantor group which is the pro-finite completion of the integers $\mathbb{Z}$. This fact has an important relationship with the *adèles* and *idèles* and its properties are the first steps in Tate's thesis.

Again, solenoids appear naturally also as basic sets of Axiom A diffeomorphisms in the sense of Smale [9]. In particular one-dimensional expanding attractors are solenoidal manifolds and were studied extensively by Bob Williams [17].

Let $\mathcal{H}(K)$ be the group of homeomorphisms of the Cantor K. Let N be a compact manifold and $\rho : \pi_1(N) \to \mathcal{H}(K)$ a homomorphism from the fundamental group of N to $\mathcal{H}(K)$. There is a lamination $\mathcal{L}_\rho$ associated to ρ called the *suspension* of ρ which is obtained by taking the quotient of $\tilde{N} \times K$ under the action of $\pi_1(N)$ given by $\gamma(x, k) = (\gamma(x), \rho(\gamma)(k))$ where $\tilde{N}$ is the universal cover of N and the action of of $\pi_1(N)$ on $\tilde{N}$ is by deck transformations.

One has a natural locally trivial fibration $p : \mathcal{L}_\rho \to N$ with fibre K.

In his paper Dennis Sullivan shows that any compact, *oriented*, 1-dimensional solenoidal manifold $\mathcal{S}$ is a mapping torus of a homeomorphism $h : K \to K$of the Cantor set K. In other words it corresponds to the representation of the fundamental group of the circle into $\mathcal{H}(K)$ induced by h. The proof is done by finding a global transversal in the oriented case. Since the topological dimension of the solenoid is one it follows that $\mathcal{S}$ embeds continuously in $\mathbb{R}^3$. However there is a nicer proof of this last fact using an unpublished idea I learned form Evgeny Shchepin.

Theorem 1. *(Shchepin) Let K be the standard triadic Cantor set in the interval $[0,1] \subset \mathbb{R} \subset \mathbb{R}^2$, and $h : K \to K$ any homeomorphism. Then h extends to a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$. We think of the Cantor set as contained in the x-axis of the (x,y)-plane.*

Proof 1. *By Tietze extension theorem the map h extends to a continuous map $f : \mathbb{R} \to \mathbb{R}$. Of course f might be neither injective nor onto. The map $F(x,y) = (x, y + f(x))$ is a homeomorphism of $\mathbb{R}^2$ to itself and $F(K)$ is the graph of h. On the other hand the map $h^{-1} : K \to K$ also extends to a map $g : \mathbb{R} \to \mathbb{R}$. The map $G(x,y) = (x - g(y), y)$ is a homeomorphism of $\mathbb{R}^2$. Thus $G \circ F$ sends K to the vertical axis: $G \circ F(x,0) = (0, h(x))$ if $x \in K$. Then we take $H = T \circ G \circ F$, where $T(x,y) = (y,x)$.* ■

Therefore we see that any oriented one dimensional solenoid $\mathcal{S}$ is contained as a "diffuse braid" in the open solid torus $\mathbb{R}^2 \times \mathbb{S}^1$ which is the mapping torus of H. In this respect one can consult [5].

The fact that any oriented one-dimensional solenoidal manifold is the suspension of a homeomorphism h of the Cantor set implies, as shown in the paper, that any such one-dimensional solenoidal manifold *is cobordant to zero*: there exists a compact two dimensional solenoidal manifold whose boundary is the given solenoidal one-dimensional manifold.

The proof is based on the fact that any homeomorphism of the Cantor set is a product of commutators and therefore there exists a representation $\rho : \pi_1(\Sigma) \to \mathcal{H}(K)$, where Σ is a smooth compact surface Σ with connected boundary a circle, such that the restriction of ρ to the element represented to the boundary is h.

The proof of the fact that de group of homeomorphisms of the Cantor set is perfect is proven in all detail in the paper [2] by R.D. Anderson.

Some of the most interesting and important solenoids are the two dimensional solenoidal manifolds (or solenoidal surfaces). In this respect Dennis himself has constructed one of the most beautiful and natural laminations whose Teichmüller space is remarkable: *The universal commensurability Teichmüller space* [11]. His paper in Acta [3], in collaboration with I. Biswas and S. Nag, is also an essential reference for this subject.

The idea of considering profinite constructions is very natural. If Σ is a compact surface and if we consider the inverse limit corresponding to the tower of all finite index coverings of Σ we obtain a two dimensional solenoidal manifold or surface lamination $\mathcal{L}$: we can consider complex structures on this lamination so that each leaf has a complex structure and the complex structures vary continuously in the transversal direction. There exists a canonical projection $\pi : \mathcal{L} \to \Sigma$. For a dense set of complex structures the restriction to each leaf is a conformal map to a finite cover of the original surface. Moreover the inverse limit of a point $K_z := \pi^{-1}\{z\},\ z \in \Sigma$ is a Cantor set. In fact in this construction one could use, to get the same inverse limit, any co-final set of finite coverings, for instance normal subgroups or even characteristic subgroups. In the latter case K_z is a nonabelian Cantor group.

The lamination $\mathcal{L}$ is the suspension of a homeomorphism $\rho : \pi_1(\Sigma) \to \mathcal{H}(K)$.

If Σ is a surface of genus two we can consider a simple closed curve γ in Σ which separates the surface into two surfaces of genus one with common boundary γ. The restriction of the lamination to γ is an oriented one dimensional solenoid. Thus there exists four homeomorphisms $f_1,\ f_2,\ g_1,\ g_2$ of the Cantor group K_z such that $[f_1, f_2] = [g_1, g_2] := h$ and the one dimensional solenoid is the suspension of h.

To me this is fascinating because these four homeomorphisms of the Cantor set satisfying the commutator relations above determine the universal solenoid. I think that it is a very interesting problem to understand the structure of these homeomorphisms.

Theorem 2 of the paper by Dennis Sullivan gives a sketch of the theorem that every solenoidal surface has a smooth structure, and in fact a laminated complex structure. Of course this is a classical theorem for surfaces (compact or not). This can be attributed to Radó and Kerékjártó since they prove that every surface can be triangulated (i.e. is homeomorphic to a simplicial complex of dimension two). There is a more recent proof of this fact by Thomassen [12]. The triangulation theorem can be adapted to solenoidal surfaces. The definition of a triangulation of a solenoidal surface is the natural one: each leaf is triangulated and the triangulation depends continuously in the transverse direction, in other words, if $L(z)$ denotes the leaf through z one requires:
For every point $z \in \mathcal{L}$ there exists a subcomplex $C \subset L(z)$ which is homeomorphic to a 2-disk and a homeomorphism $\phi : C \times K \to \mathcal{L}$ such that ϕ restricted to $C \times \{k\}$ is a simplicial linear homeomorphsm from $C \times \{k\}$ onto a subcomplex of the triangulated leaf

$$L(\phi(c, k)) \ (c \in C, \ k \in K).$$

Theorem 2. *Let $\mathcal{L}$ be a topological compact solenoidal surface then $\mathcal{L}$ can be triangulated.*

Let me give a sketch of my own proof of this theorem. The Riemann mapping theorem together with Carathéodory's theorem of prime ends imply that any continuous Jordan curve in the plane is locally flat, which implies that every Jordan curve has a topological tubular neighborhood. It is easy to prove - via the Riemann mapping theorem - that given two topological disks which are the images of two topological embeddings $\phi_i : \bar{\Delta} \to S$ $(i = 1, 2)$ of the unit closed disk in the complex plane into a topological surface S one can perturb ϕ_i $(i = 1, 2)$ to two embeddings ϕ_i such that the images of $\mathbb{S}^1 = \partial\bar{\Delta}$ meet topologically transversally (locally like the intersection of the coordinate axis in $\mathbb{R}^2$ at the origin). A Riemann surface can be covered by coordinate charts $\psi_j : \bar{\Delta} \to S$ such that the union of images of the disk of radius $1/2$ still cover the surface and the covering is locally finite. We can perturb slightly the embeddings so that the images of the boundary of the disk of radius $1/2$ meet topologically transversally. The union of the images of the these boundary circles divide the surface into cells with boundary a Jordan curve with a finite number of marked points where two such curves meet transversally. Using these points we can subdivide each cell to triangulate the Riemann surface. For a solenoidal surface $\mathcal{L}$ a similar construction works: we can cover the lamination with laminated charts $f_i : \bar{\Delta} \times K \to \mathcal{L}$ and then we can perturb these charts to have in each leaf a situation like the previous for a Riemann surface.

A triangulated solenoidal surface has a natural flat structure with singularities: we give each triangle of the triangulation the euclidean metric so that it is an equilateral triangle and all of these triangles have edges of equal lengths. This provides each leaf with a flat metric singular at the vertices (a sort of laminated Veech surface). By Riemann extension theorem each leaf is a complex surface and thus each solenoidal surface has a complex structure

Reciprocally every compact smooth solenoidal manifolds $\mathcal{S}$ has a triangulation à la Cairns. Let me give a sketch of the proof which is modeled on Cairns proof. Whitney embedding theorem is valid for smooth solenoidal manifolds: there exists a topological embedding $j : \mathcal{S} \to \mathbb{R}^n$. This follows from the usual fact that smooth real valued functions (in the sense of laminations) separate points. The embedding j when restricted to a leaf is an embedding (not necessarely a proper embedding) and if $\Phi : \mathbb{D}^k \times T \to \mathcal{S}$ (T a closed subset of the Cantor set) is a solenoidal chart the composition $j \circ \phi$ when restricted to a plaque $\mathbb{D}^k \times \{t\}$, $t \in T$ is an embedding $k_t : \mathbb{D}^k \to \mathbb{R}^n$.

We require that the embeddings of plaques depend continuously on the transverse parameter (i.e. the map $t \mapsto k_t \in \mathrm{C}^\infty(\mathbb{D}^k, \mathbb{R}^n)$ is continuous). Then if we consider the solenoidal manifold $j(\mathcal{S}) \subset \mathbb{R}^n$ we can apply a very large homothetic transformation$T$, $x \mapsto rx$ $x \in \mathbb{R}^n$, $r \in \mathbb{R}$ with $r > 0$ very large so that the curvature of the leaves of $j(\mathcal{S}$ is almost zero). Now we consider the canonical cubulation by unit cubes of $\mathcal{R}^n$ and the intersection of $T(j(\mathcal{S}))$ with each cube of the cubulation. Since we can assume without difficulty that $J(\mathcal{S})$ is transverse to all the skeletons of the cubulation, we see that each leaf is almost an affine subspace of dimension k with respect to a unit cube, so that each leaf meets each cube in a convex polytope of dimension k after subdividing in an obvious way each of these polytopes we get the triangulation of the solenoidal manifold.

Since every solenoidal surface has a smooth structure we can provide each leaf with a Riemannian metric in such a way that the metric is smooth on each leaf and it depends continuously on the transverse parameter. We call such a solenoidal surface with a leaf-wise metric metric g a *solenoidal Riemannian surface* $(\mathcal{S}, g)$.

Given a *compact* solenoidal surface $(\mathcal{S}, g)$ we see that each leaf has a conformal type with respect to g, i.e for any $z \in \mathcal{S}$ the universal covering of the leaf $L(z)$ is conformally equivalent to the Riemann sphere (elliptic leaf) the complex plane (parabolic leaf) or the Poincaré disk (hyperbolic leaf). If g' es any other leaf-wise smooth Riemannian metric the conformal type of the leaf does not change. This is a beautiful observation of Elmar Winkelnkemper (1976). Therefore one can speak of a *hyperbolic solenoidal Riemannian surface* when all the leaves are of hyperbolic type. We have the analog of the uniformization theorem of Koebe-Poincaré for compact hyperbolic solenoidal Riemannian surface.

Theorem 3. *(Candel [4] and Verjovsky [14]). If every leaf of a laminar Riemannian surface is conformally covered by the disk, then the unique constant curvature minus one metric on each leaf is transversally continuous.*

Sullivan states and sketches a proof of the following theorem of Alberto Candel [4]:

Theorem 4. *For any transversally continuous Riemannian metric on a smooth laminar surface, sometimes both but at least one of the following holds:*

(1) *The universal cover of every leaf is conformally the disk.*
(2) *There is a nontrivial tranversal measure (a measure on each transversal so that the germs of transversal holonomy maps along paths are measure preserving).*

Of course there are compact solenoidal surfaces such that every leaf has universal covering conformally equivalent to the euclidean plane. For instance the inverse limit of finite covers of a flat 2-torus. For these laminations some times it is impossible to simultaneously uniformize all the leaves [6].

Sullivan gives an example of a noncompact surface lamination without transverse measure but there is, in my opinion, a better *compact* example which of course Dennis knows since I learned it from him. Let $\mathcal{S}_2$ be the dyadic solenoid given as the inverse limit of

$$\cdots \longrightarrow \mathbb{S}^1 \xrightarrow{z \to z^2} \mathbb{S}^1 \xrightarrow{z \to z^2} \mathbb{S}^1.$$

Then $\mathcal{S}_2$ is a compact abelian solenoidal group with a canonical metric which induces Haar measure on the group. After choosing an orientation, there is a unit vector field Y tangent to the lamination. The squaring map $F(Z) = Z^2$ is an isomorphism of $\mathcal{S}_2$ onto itself. Its derivative in the sense of laminations expands by two every unit tangent vector $Y(x)$. The suspension of F is defined as the mapping torus of F. It is a two dimensional lamination. There is the canonical suspension flow generated by the vector field X tangent to $\mathcal{S}_2$. In fact the leaves of this lamination are the orbits of a locally free action of the real affine group since we have the Lie bracet relation $[X, Y] = Y$. It is not difficult to prove:

Proposition 1. *If $\mathcal{L}$ is a compact lamination whose leaves are given by a locally free action of the real affine group, then the lamination does not admit a transverse measure.*

The last part of the paper deals with the Teichmüller theory of compact solenoidal (or laminar) surfaces. For a compact laminar surface such that all its leaves are hyperbolic it is possible to develop Teichmüller theory. Almost everything valid for a hyperbolic Riemann surface is also valid for such a lamination. In general the Teichmüller space is infinite dimensional if the transverse structure is a Cantor set.

Thus it is possible to speak of Teichmüller distance, quadratic differentials, etc.

Theorem 5. *The space of hyperbolic structures on a hyperbolic laminar surface (as in Theorem 4) up to isometries isotopic to the identity has the structure of a separable complex Banach manifold. The metric is the natural Teichmüller metric based on the minimal conformal distortion of a map between structures. The isotopy classes of homeomorphisms preserving a chosen leaf act by isometries on this Banach manifold.*

As was remarked before, Sullivan constructs the universal Teichmüller space of the solenoidal surface $\mathcal{S}$ obtained by taking the inverse limit of all finite pointed covers of a compact surface of genus greater than one and chosen base point. The base points upstairs in the covers determine a point and a distinguished leaf L in the inverse limit solenoidal surface. In this space the commensurability automorphism group of the fundamental group of any higher genus compact surface acts by isometries. This group is independent of the genus dy definition.

Theorem 6. *The space of hyperbolic structures up to isometry preserving the distinguished leaf on this solenoidal surface $\mathcal{S}$ is non Hausdorff and any Hausdorff quotient is a point.*

The proof ot this result relies on the recent deep results by Jeremy Kahn and Vladimir Marković on the validity of the Ehrenpreis Conjecture [7].

The remark by Sullivan is that the action of the commensurability automorphism group of the fundamental group is by isometries and minimal. The action is described in the paper in *Acta Mathematica* [3] mentioned before.

Sullivan does not include in his article the role of laminations in holomorphic dynamics, a subject created by him to prove the Feiganbaum universality conjectures, and continued, for instance, in the use of 3-dimensional hyperbolic laminations by Misha Lyubich and Yair Minsky. in [8].

Given any compact manifold M a representation of $\rho : \pi_1(M) \to \mathcal{H}(K)$, where $\mathcal{H}(K)$ is the group of homeomorphisms of the Cantor set, gives rise to a solenoidal manifold. Therefore if M is any compact manifold with residually finite fundamental group (as in the case of a Riemann surface of genus bigger than one or any compact hyperbolic manifold) one has a lamination by considering the inverse limit of the tower of its finite covers. This is, in a sense, the *profinite completion* of a manifold with residually finite fundamental group. The fundamental groups of compact hyperbolic 3-manifolds are residually finite so that we can consider the infinite tower of finite covers.

A direct consequence of the recent results by Ian Agol [1] and Daniel Wise [17] which solve in the affirmative the question by Bill Thurston whether every hyperbolic 3-manifold M virtually fibers over the circle (i.e. there exists a finite covering $\tilde{M}$ and a locally-trivial fibration over the circle $p : \tilde{M} \to \mathbb{S}^1$) we have:

Theorem 7. *Let M be a compact hyperbolic 3-manifold and let $\mathcal{L}(M)$ be the compact 3-dimensional lamination obtained by the inverse limit of the directed set of its finite covers. Then:*

1. *$\mathcal{L}(M)$ fibers over M with fiber the Cantor set*
2. *There exists a locally trivial fibration $\pi : \mathcal{L}(M) \to \mathbb{S}^1$ with fiber a laminar surface $\mathcal{S}$.*
3. *By 2. there exists a homeomorphism $f : \mathcal{S} \to \mathcal{S}$ such that $\mathcal{L}(M)$ is obtained by suspending f.*

I think that the study of the homeomorphism f in *3* above is interesting. It is the lifting, in the tower of coverings of the fibre $p^{-1}(\{1\})$ of the virtual fibration, of the pseudo-Anosov homeomorphism of the fibre which determines the fibration over the circle.

A solenoidal manifold (or lamination) is said to be hyperbolic if there exist a Riemannian metric for which every leaf has constant negative curvature -1.

In view of theorem 7 some natural questions arise:

Question. *Let $\mathcal{L}$ be a compact laminar surface. Let $f : \mathcal{L} \to \mathcal{L}$ be a homeomorphism. Let M be the 3-dimensional compact solenoidal manifold which is obtained by suspending f.*

1. *When is M a hyperbolic compact solenoidal 3-manifold ?*
2. *Is there a classification à la Thurston of isotopy classes of homeomorphisms of compact laminar surfaces?*
3. *Does every compact hyperbolic* 3*-dimensional hyperbolic lamination fibers over the circle?*

Another topic would be to develop the theory of *geodesic laminations* for compact hyperbolic solenoidal surfaces.

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