

ORBIFOLD JACOBIAN ALGEBRAS FOR INVERTIBLE POLYNOMIALS

ALEXEY BASALAEV, ATSUSHI TAKAHASHI, AND ELISABETH WERNER

ABSTRACT. An important invariant of a polynomial f is its Jacobian algebra defined by its partial derivatives. Let f be invariant with respect to the action of a finite group of diagonal symmetries G . We axiomatically define an orbifold Jacobian $\mathbb{Z}/2\mathbb{Z}$ -graded algebra for the pair (f, G) and show its existence and uniqueness in the case, when f is an invertible polynomial. In case when f defines an ADE singularity, we illustrate its geometric meaning.

1. INTRODUCTION

Let $f = f(\mathbf{x}) = f(x_1, \dots, x_N) \in \mathbb{C}[x_1, \dots, x_N]$ be a polynomial such that the *Jacobian algebra* $\text{Jac}(f) := \mathbb{C}[x_1, \dots, x_N]/(\partial f/\partial x_1, \dots, \partial f/\partial x_N)$ of f is a finite dimensional \mathbb{C} -algebra. In this paper, we shall give axioms which should characterize a generalization of the Jacobian algebra $\text{Jac}(f)$ of f for the pair (f, G) where G is a finite abelian group acting diagonally on variables which respects f .

Such a pair (f, G) , often called a *Landau-Ginzburg orbifold*, has been studied intensively by many mathematicians and physicists working in mirror symmetry for more than twenty years since it yields important, interesting and unexpected geometric information. In particular, the so-called orbifold construction of a mirror manifold from a Calabi-Yau hypersurface is very important.

Certain works towards the definition of the Frobenius algebras associated to the pair (f, G) were also done previously by R. Kaufmann and M. Krawitz. In [K03], R. Kaufmann proposes a general construction of the *orbifolded Frobenius superalgebra* of (f, G) . In order to build such a $\mathbb{Z}/2\mathbb{Z}$ -graded algebra, one should make a certain non-unique choice called the “choice of a two cocycle”. A different choice of this cocycle gives indeed a different product structure. This construction was later used by Kaufmann in [K06] for mirror symmetry purposes from the point of view of physics. In [Kr], M. Krawitz proposes a very special construction of a commutative (not a $\mathbb{Z}/2\mathbb{Z}$ -graded) algebra, for the pair (f, G) . Later enhanced in [FJJS] this definition was used to set up mirror symmetry on the level of Frobenius algebras. However, the crucial part of it remained the particularly fixed product that could only be well-defined for weighted-homogeneous polynomials. There is also no explanation why a particular product structure is chosen.

Mirror symmetry on the level of Frobenius algebras is a first step towards the mirror isomorphism of Frobenius manifolds where the key role is played by the so-called *primitive form*. From the point of view of mirror symmetry, the algebras we consider here are those on the complex geometry side, the so-called B-model side. The major advantage of our work comparing to that of Kaufmann and Krawitz is that our construction works as a starting point for the mirror symmetry on the level of Frobenius manifolds having the notion of a primitive form (cf. [S1, S2, ST]) in the definition (cf. the role of ζ in Definition 16). The second important point is the following. In both Kaufmann and Krawitz constructions, one predefines the product either by the definition or by the special choice of a two cocycle. In our axiomatization, we do not do this and hence we are able to study our algebras even for non-weighted homogeneous polynomials. We hope to address this question later. The last but not least is that our algebra inherits a natural

$\mathbb{Z}/2\mathbb{Z}$ -grading from the Hodge theory associated to (f, G) . This $\mathbb{Z}/2\mathbb{Z}$ -grading appears only in an abstract way in the definition of Kaufmann and was not considered at all by Krawitz.

Let us comment in detail on the first point. It is well-known that $\text{Jac}(f)$ has a structure of a Frobenius algebra (see [AGV85]). Namely, by a choice of a nowhere vanishing holomorphic N -form, there is an isomorphism $\text{Jac}(f) \cong \Omega_f := \Omega^N(\mathbb{C}^N)/(df \wedge \Omega^{N-1}(\mathbb{C}^N))$. It is on Ω_f , where a natural or canonical non-degenerate symmetric bilinear form, called *the residue pairing*, exists. As a result, the above isomorphism equips $\text{Jac}(f)$ with an induced bilinear form. Therefore, even if the group G is trivial, it is important to consider a pair $(\text{Jac}(f), \Omega_f)$. This pair can also be considered as an example of a pair $(HH^\bullet(\mathcal{C}), HH_\bullet(\mathcal{C}))$, consisting of the Hochschild cohomology and the Hochschild homology of a suitable dg- or \mathcal{A}_∞ -category, which has a rich algebraic structure.

In this paper, we shall first introduce a G -twisted version of the vector space Ω_f , which is denoted by $\Omega'_{f,G}$. This is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space, which also has a G -grading, with a natural non-degenerate bilinear form called the *orbifold residue pairing*, a natural generalization of the residue pairing on Ω_f . Then, the G -twisted version of the Jacobian algebra, denoted by $\text{Jac}'(f, G)$, will be introduced axiomatically as a part of the structure of the pair $(\text{Jac}'(f, G), \Omega'_{f,G})$ in the way it is in the classical situation when the group G is trivial. As a result, the algebra $\text{Jac}'(f, G)$ inherits many structures, defined naturally on $\Omega'_{f,G}$, such as a $\mathbb{Z}/2\mathbb{Z}$ -grading, a G -grading, equivariance with respect to automorphisms of the pair (f, G) , the orbifold residue pairing and so on. Our axiomatization of a G -twisted Jacobian algebra lists a minimum conditions to be satisfied, in particular, we do not prescribe the product structure.

The expected Jacobian algebra $\text{Jac}(f, G)$ for the pair (f, G) , which we shall call *the orbifold Jacobian algebra* of (f, G) , will be given as the G -invariant subalgebra of the G -twisted version $\text{Jac}'(f, G)$. However, it is not clear in general whether such an algebra as $\text{Jac}'(f, G)$ exists or not. Even if it exists it may not be unique.

The main result of this paper is the existence and the uniqueness of the G -twisted Jacobian algebra $\text{Jac}'(f, G)$ for an invertible polynomial f with a subgroup G of the maximal abelian symmetry group G_f (Theorem 22). Namely, it is uniquely determined up to isomorphism by our axiomatization. Moreover, we show that if G is a subgroup of $\text{SL}(N; \mathbb{C})$ then the orbifold Jacobian algebra $\text{Jac}(f, G)$ has a structure of a $\mathbb{Z}/2\mathbb{Z}$ -graded commutative Frobenius algebra.

Another interesting result of ours (Theorem 64) concerns the case when f is an invertible polynomial giving a singularity of ADE-type and G is a subgroup of $G_f \cap \text{SL}(N; \mathbb{C})$. We show that in this case our orbifold Jacobian algebra $\text{Jac}(f, G)$ is isomorphic to the usual Jacobian algebra. This result complies with the results of [ET13a], where concerning a crepant resolution $\widehat{\mathbb{C}^3}/G$ of \mathbb{C}^3/G , it is shown that the geometry of vanishing cycles for a holomorphic map $\widehat{f} : \widehat{\mathbb{C}^3}/G \rightarrow \mathbb{C}$ associated to f is equivalent to the one for the polynomial \widehat{f} . Therefore, our orbifold Jacobian algebra is not only natural from the view point of algebra but also from the view point of geometry.

Remark 1. After the the current paper was put on arXiv, the certain progress was made in understanding the orbifold Jacobian algebra. In particular, based on the results of the current paper, it was shown in [BT] that for f invertible the algebra $\text{Jac}(f, G)$ is isomorphic to the Hochschild cohomology of the category of G -equivariant matrix factorizations of f . Moreover, some applications to the more classical problems of singularity theory were found in [BTW2].

Acknowledgements. The first named author was supported by International Laboratory of Cluster Geometry NRU HSE, RF Government grant, ag. no. 075-15-2021-608 dated 08.06.2021. The second named author was supported by JSPS KAKENHI Grant Number JP16H06337, JP26610008. We are grateful to Wolfgang Ebeling for fruitful discussions.

We are also very grateful to an anonymous referee for many comments and remarks.

2. PRELIMINARIES

Definition 2. Let n be a non-negative integer and $f = f(\mathbf{x}) = f(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ a polynomial.

(i) The *Jacobian algebra* $\text{Jac}(f)$ of f is the \mathbb{C} -algebra defined as

$$\text{Jac}(f) = \mathbb{C}[x_1, \dots, x_n] / \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right). \quad (2.1)$$

If $\text{Jac}(f)$ is a finite-dimensional \mathbb{C} -algebra, then set $\mu_f := \dim_{\mathbb{C}} \text{Jac}(f)$ and call it the *Milnor number* of f . In particular, if $n = 0$ then $\text{Jac}(f) = \mathbb{C}$ and $\mu_f = 1$.

(ii) The *Hessian* of f is defined as

$$\text{hess}(f) := \det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1,\dots,n}. \quad (2.2)$$

In particular, if $n = 0$ then $\text{hess}(f) = 1$.

Throughout this paper, we denote by N a positive integer and by

$$f = f(\mathbf{x}) = f(x_1, \dots, x_N) \in \mathbb{C}[x_1, \dots, x_N]$$

a polynomial such that the Jacobian algebra of f is a finite-dimensional \mathbb{C} -algebra, unless otherwise stated.

Let $\Omega^p(\mathbb{C}^N)$ be the \mathbb{C} -module of regular p -forms on \mathbb{C}^N . Consider the \mathbb{C} -module

$$\Omega_f := \Omega^N(\mathbb{C}^N) / (df \wedge \Omega^{N-1}(\mathbb{C}^N)). \quad (2.3)$$

Note that Ω_f is naturally a free $\text{Jac}(f)$ -module of rank one, namely, by choosing a nowhere vanishing N -form $\tilde{\omega} \in \Omega^N(\mathbb{C}^N)$ we have the following isomorphism

$$\text{Jac}(f) \xrightarrow{\cong} \Omega_f, \quad [\phi(\mathbf{x})] \mapsto [\phi(\mathbf{x})\omega] := [\phi(\mathbf{x})\tilde{\omega}], \quad (2.4)$$

where $\omega := [\tilde{\omega}]$ is the residue class of $\tilde{\omega}$ in Ω_f .

Remark 3. Such a class $\omega \in \Omega_f$ giving the isomorphism (2.4) is a non-zero constant multiple of the residue class of $dx_1 \wedge \dots \wedge dx_N$.

Proposition 4 (cf. Section I.5.11 [AGV85]). *Define a \mathbb{C} -bilinear form $J_f : \Omega_f \otimes_{\mathbb{C}} \Omega_f \rightarrow \mathbb{C}$ as*

$$J_f(\omega_1, \omega_2) := \text{Res}_{\mathbb{C}^N} \left[\begin{array}{c} \phi(\mathbf{x})\psi(\mathbf{x})dx_1 \wedge \dots \wedge dx_N \\ \frac{\partial f}{\partial x_1} \dots \frac{\partial f}{\partial x_N} \end{array} \right], \quad (2.5)$$

where $\omega_1 = [\phi(\mathbf{x})dx_1 \wedge \dots \wedge dx_N]$ and $\omega_2 = [\psi(\mathbf{x})dx_1 \wedge \dots \wedge dx_N]$. Then, the bilinear form J_f on Ω_f is non-degenerate. Moreover, for $\phi(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_N]$,

$$J_f([\phi(\mathbf{x})dx_1 \wedge \dots \wedge dx_N], [\text{hess}(f)dx_1 \wedge \dots \wedge dx_N]) \neq 0 \quad (2.6)$$

if and only if $\phi(\mathbf{0}) \neq 0$. In particular, we have

$$J_f([dx_1 \wedge \dots \wedge dx_N], [\text{hess}(f)dx_1 \wedge \dots \wedge dx_N]) = \mu_f. \quad (2.7)$$

Under the isomorphism (2.4), the residue pairing endows the Jacobian algebra $\text{Jac}(f)$ with a structure of a Frobenius algebra.

Definition 5. An associative \mathbb{C} -algebra (A, \circ) is called *Frobenius* if there is a non-degenerate bilinear form $\eta : A \otimes A \rightarrow \mathbb{C}$ such that $\eta(X \circ Y, Z) = \eta(X, Y \circ Z)$ for $X, Y, Z \in A$.

Definition 6. The group of maximal diagonal symmetries of f is defined as

$$G_f := \{(\lambda_1, \dots, \lambda_N) \in (\mathbb{C}^*)^N \mid f(\lambda_1 x_1, \dots, \lambda_N x_N) = f(x_1, \dots, x_N)\}. \quad (2.8)$$

We shall always identify G_f with the subgroup of diagonal matrices of $\mathrm{GL}(N; \mathbb{C})$. Set

$$G_f^{\mathrm{SL}} := G_f \cap \mathrm{SL}(N; \mathbb{C}). \quad (2.9)$$

Remark 7. For a finite subgroup $G \subseteq G_f$, the pair (f, G) is often called a *Landau-Ginzburg orbifold*.

From now on, we shall denote by G a finite subgroup of G_f unless otherwise stated. In what follows, define also $\mathbf{e}[\alpha] := \exp(2\pi\sqrt{-1}\alpha)$. The group G acts naturally on \mathbb{C}^N and each element $g \in G$ has a unique expression of the form

$$g = \mathrm{diag} \left(\mathbf{e} \left[\frac{a_1}{r} \right], \dots, \mathbf{e} \left[\frac{a_N}{r} \right] \right) \quad \text{with } 0 \leq a_i < r, \quad (2.10)$$

where r is the order of g . We use the notation $(a_1/r, \dots, a_N/r)$ or $\frac{1}{r}(a_1, \dots, a_N)$ for the element g . The *age* of g , which is introduced in [IR, V], is defined as the rational number

$$\mathrm{age}(g) := \frac{1}{r} \sum_{i=1}^N a_i. \quad (2.11)$$

Note that if $g \in G_f^{\mathrm{SL}}$ then $\mathrm{age}(g) \in \mathbb{Z}$.

For each $g \in G$, we denote by $\mathrm{Fix}(g) := \{\mathbf{x} \in \mathbb{C}^N \mid g \cdot \mathbf{x} = \mathbf{x}\}$ the fixed locus of g , by $N_g := \dim_{\mathbb{C}} \mathrm{Fix}(g)$ its dimension and by $f^g := f|_{\mathrm{Fix}(g)}$ the restriction of f to the fixed locus of g . Note that since G acts diagonally on \mathbb{C}^N , $\mathrm{Fix}(g)$ is a linear subspace of \mathbb{C}^N .

Proposition 8 (cf. Proposition 5 in [ET13b]). *For each $g \in G$, we have a natural surjective \mathbb{C} -algebra homomorphism $\mathrm{Jac}(f) \rightarrow \mathrm{Jac}(f^g)$. In particular, the Jacobian algebra $\mathrm{Jac}(f^g)$ is also finite dimensional.*

Proof. We may assume that $\mathrm{Fix}(g) = \{\mathbf{x} \in \mathbb{C}^N \mid x_{N_g+1} = \dots = x_N = 0\}$ by a suitable renumbering of indices. Since f is invariant under G , $g \cdot x_i \neq x_i$ for $i = N_g + 1, \dots, N$ and $\frac{\partial f}{\partial x_{N_g+1}}, \dots, \frac{\partial f}{\partial x_N}$ form a regular sequence, we have

$$\left(\frac{\partial f}{\partial x_{N_g+1}}, \dots, \frac{\partial f}{\partial x_N} \right) \subset (x_{N_g+1}, \dots, x_N).$$

Therefore, we have a natural surjective \mathbb{C} -algebra homomorphism

$$\begin{aligned} \mathrm{Jac}(f) &= \mathbb{C}[x_1, \dots, x_N] \Big/ \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N} \right) \\ &\rightarrow \mathbb{C}[x_1, \dots, x_N] \Big/ \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{N_g}}, x_{N_g+1}, \dots, x_N \right) \\ &= \mathbb{C}[x_1, \dots, x_{N_g}] \Big/ \left(\frac{\partial f^g}{\partial x_1}, \dots, \frac{\partial f^g}{\partial x_{N_g}} \right) = \mathrm{Jac}(f^g). \end{aligned}$$

□

Corollary 9. *For each $g \in G$, Ω_{f^g} is naturally equipped with a structure of $\mathrm{Jac}(f)$ -module.*

Proof. Since Ω_{f^g} is a free $\mathrm{Jac}(f^g)$ -module of rank one (cf. (2.4)), the surjective \mathbb{C} -algebra homomorphism $\mathrm{Jac}(f) \rightarrow \mathrm{Jac}(f^g)$ yields the statement. □

3. ORBIFOLD JACOBIAN ALGEBRAS

In order to introduce an orbifold Jacobian algebra of the pair (f, G) , we first define axiomatically a G -twisted Jacobian algebra of f .

3.1. Setup.

Definition 10. Define a $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -module $\Omega'_{f,G} = (\Omega'_{f,G})_{\bar{0}} \oplus (\Omega'_{f,G})_{\bar{1}}$, $\bar{i} \in \mathbb{Z}/2\mathbb{Z}$, by

$$(\Omega'_{f,G})_{\bar{0}} := \bigoplus_{\substack{g \in G \\ N - N_g \equiv 0 \pmod{2}}} \Omega'_{f,g}, \quad (\Omega'_{f,G})_{\bar{1}} := \bigoplus_{\substack{g \in G \\ N - N_g \equiv 1 \pmod{2}}} \Omega'_{f,g}, \quad (3.1a)$$

$$\Omega'_{f,g} := \Omega_{f^g}. \quad (3.1b)$$

Here, for each $g \in G$ with $\text{Fix}(g) = \{\mathbf{0}\}$,

$$\Omega_{f^g} = \Omega^0(\{\mathbf{0}\}) / (df^g \wedge \Omega^{-1}(\{\mathbf{0}\})) = \Omega^0(\{\mathbf{0}\}) \quad (3.1c)$$

is the \mathbb{C} -module of rank one consisting of constant functions on $\{\mathbf{0}\}$.

Since the group G acts on each Ω_{f^g} by the pull-back of forms via its action on $\text{Fix}(g)$, we can define the following $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -module.

Definition 11. Define a $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -module $\Omega_{f,G}$ as the G -invariant part of $\Omega'_{f,G}$,

$$\Omega_{f,G} = (\Omega'_{f,G})^G. \quad (3.2)$$

That is, $\Omega_{f,G} = (\Omega_{f,G})_{\bar{0}} \oplus (\Omega_{f,G})_{\bar{1}}$, $\bar{i} \in \mathbb{Z}/2\mathbb{Z}$, is given by

$$(\Omega_{f,G})_{\bar{0}} := \bigoplus_{\substack{g \in G \\ N - N_g \equiv 0 \pmod{2}}} \Omega_{f,g}, \quad (\Omega_{f,G})_{\bar{1}} := \bigoplus_{\substack{g \in G \\ N - N_g \equiv 1 \pmod{2}}} \Omega_{f,g}, \quad (3.3a)$$

$$\Omega_{f,g} := (\Omega_{f^g})^G. \quad (3.3b)$$

Definition 12. Define a non-degenerate \mathbb{C} -bilinear form $J_{f,G} : \Omega'_{f,G} \otimes_{\mathbb{C}} \Omega'_{f,G} \rightarrow \mathbb{C}$, called the *orbifold residue pairing*, by

$$J_{f,G} := \bigoplus_{g \in G} J_{f,g}, \quad (3.4a)$$

where $J_{f,g}$ is a perfect \mathbb{C} -bilinear form $J_{f,g} : \Omega'_{f,g} \otimes_{\mathbb{C}} \Omega'_{f,g^{-1}} \rightarrow \mathbb{C}$ defined by

$$J_{f,g}(\omega_1, \omega_2) := (-1)^{N - N_g} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g) \right] \cdot |G| \cdot \text{Res}_{\text{Fix}(g)} \left[\begin{array}{c} \phi \psi dx_{i_1} \wedge \cdots \wedge dx_{i_{N_g}} \\ \frac{\partial f^g}{\partial x_{i_1}} \cdots \frac{\partial f^g}{\partial x_{i_{N_g}}} \end{array} \right] \quad (3.4b)$$

for $\omega_1 = [\phi dx_{i_1} \wedge \cdots \wedge dx_{i_{N_g}}] \in \Omega'_{f,g}$ and $\omega_2 = [\psi dx_{i_1} \wedge \cdots \wedge dx_{i_{N_g}}] \in \Omega'_{f,g^{-1}}$, where $x_{i_1}, \dots, x_{i_{N_g}}$ are coordinates of $\text{Fix}(g)$. For each $g \in G$ with $\text{Fix}(g) = \{\mathbf{0}\}$, we define

$$J_{f,g}(1_g, 1_{g^{-1}}) := (-1)^N \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g) \right] \cdot |G|, \quad (3.4c)$$

where $1_g \in \Omega'_{f,g}$ and $1_{g^{-1}} \in \Omega'_{f,g^{-1}}$ denote the constant functions on $\{\mathbf{0}\}$ whose values are 1.

Proposition 13. *The \mathbb{C} -bilinear form $J_{f,G}$ is G -twisted $\mathbb{Z}/2\mathbb{Z}$ -graded symmetric in the sense that*

$$J_{f,G}(\omega_1, \omega_2) = (-1)^{N - N_g} \cdot \mathbf{e}[-\text{age}(g)] \cdot J_{f,G}(\omega_2, \omega_1) \quad (3.5)$$

for $\omega_1 \in \Omega'_{f,g}$ and $\omega_2 \in \Omega'_{f,g^{-1}}$.

Proof. Let the notations be as in Definition 12. Since $\text{Fix}(g) = \text{Fix}(g^{-1})$, $f^g = f^{g^{-1}}$ and $\text{age}(g) + \text{age}(g^{-1}) = N - N_g$, we have

$$\begin{aligned} J_{f,G}(\omega_1, \omega_2) &= J_{f,g}(\omega_1, \omega_2) \\ &= \mathbf{e} \left[-\frac{1}{2} \text{age}(g) + \frac{1}{2} \text{age}(g^{-1}) \right] \cdot J_{f,g^{-1}}(\omega_2, \omega_1) \\ &= (-1)^{N-N_g} \cdot \mathbf{e}[-\text{age}(g)] \cdot J_{f,G}(\omega_2, \omega_1). \end{aligned}$$

□

For a \mathbb{C} -algebra R , denote by $\text{Aut}_{\mathbb{C}\text{-alg}}(R)$ the group of all \mathbb{C} -algebra automorphisms of R . Note that G is naturally identified with a subgroup of $\text{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1, \dots, x_N])$.

Definition 14. Define the group $\text{Aut}(f, G)$ of automorphisms of (f, G) as

$$\text{Aut}(f, G) := \{ \varphi \in \text{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1, \dots, x_N]) \mid \varphi(f) = f, \varphi \circ g \circ \varphi^{-1} \in G \text{ for all } g \in G \}.$$

It is obvious that G is naturally identified with a subgroup of $\text{Aut}(f, G)$. Note that a \mathbb{C} -algebra automorphism $\varphi \in \text{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1, \dots, x_N])$ is G -equivariant if and only if $\varphi \circ g \circ \varphi^{-1} = g$ for all $g \in G$.

Remark 15. Let $\mathbb{C}[x_1, \dots, x_N] * G$ be the skew group ring which is a \mathbb{C} -vector space $\mathbb{C}[x_1, \dots, x_N] \otimes_{\mathbb{C}} \mathbb{C}[G]$ with a product defined as $(\phi_1 \otimes g_1)(\phi_2 \otimes g_2) = (\phi_1 g_1(\phi_2)) \otimes g_1 g_2$ for any $\phi_1, \phi_2 \in \mathbb{C}[x_1, \dots, x_N]$ and $g_1, g_2 \in G$. Then the group $\text{Aut}(f, G)$ can be regarded as the subgroup of all $\varphi' \in \text{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1, \dots, x_N] * G)$ such that $\varphi'(f \otimes \text{id}) = f \otimes \text{id}$. For $\varphi \in \text{Aut}(f, G)$, the correspondence element in $\text{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1, \dots, x_N] * G)$ is given by $\phi \otimes g \mapsto \varphi(\phi) \otimes (\varphi \circ g \circ \varphi^{-1})$.

An element $\varphi \in \text{Aut}(f, G)$ regarded as a bi-regular map on \mathbb{C}^N maps $\text{Fix}(\varphi \circ g \circ \varphi^{-1})$ to $\text{Fix}(g)$ for each $g \in G$. Hence, the group $\text{Aut}(f, G)$ acts naturally on $\Omega'_{f,G}$ by

$$\Omega'_{f,g} \longrightarrow \Omega'_{f, \varphi \circ g \circ \varphi^{-1}}, \quad \omega \mapsto \varphi^*|_{\text{Fix}(g)} \omega, \quad (3.6)$$

where $\varphi^*|_{\text{Fix}(g)}$ denotes the restriction of the pullback φ^* of differential forms to $\text{Fix}(g)$. In order to simplify the notation, for each $\varphi \in \text{Aut}(f, G)$, we shall denote by φ^* the action of φ on $\Omega'_{f,G}$. It also follows that $\text{Aut}(f, G)$ acts naturally on $\Omega_{f,G}$.

3.2. Axioms.

Definition 16. A G -twisted Jacobian algebra of f is a $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -algebra

$$\text{Jac}'(f, G) = \text{Jac}'(f, G)_{\bar{0}} \oplus \text{Jac}'(f, G)_{\bar{1}},$$

$\bar{i} \in \mathbb{Z}/2\mathbb{Z}$, satisfying the following axioms:

- (i) For each $g \in G$, there is a \mathbb{C} -module $\text{Jac}'(f, g)$ isomorphic to $\Omega'_{f,g}$ as a \mathbb{C} -module satisfying the following conditions:
 - (a) For the identity id of G ,

$$\text{Jac}'(f, \text{id}) = \text{Jac}(f). \quad (3.7)$$

- (b) We have

$$\text{Jac}'(f, G)_{\bar{0}} = \bigoplus_{\substack{g \in G \\ N - N_g \equiv 0 \pmod{2}}} \text{Jac}'(f, g), \quad (3.8a)$$

$$\text{Jac}'(f, G)_{\bar{1}} = \bigoplus_{\substack{g \in G \\ N - N_g \equiv 1 \pmod{2}}} \text{Jac}'(f, g). \quad (3.8b)$$

- (ii) The $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -algebra structure \circ on $\text{Jac}'(f, G)$ satisfies

$$\text{Jac}'(f, g) \circ \text{Jac}'(f, h) \subset \text{Jac}'(f, gh), \quad g, h \in G, \quad (3.9)$$

and the \mathbb{C} -subalgebra $\text{Jac}'(f, \text{id})$ of $\text{Jac}'(f, G)$ coincides with the \mathbb{C} -algebra $\text{Jac}(f)$.

- (iii) The $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -algebra $\text{Jac}'(f, G)$ is such that the \mathbb{C} -module $\Omega'_{f, G}$ has a structure of a $\text{Jac}'(f, G)$ -module

$$\vdash: \text{Jac}'(f, G) \otimes \Omega'_{f, G} \longrightarrow \Omega'_{f, G}, \quad X \otimes \omega \mapsto X \vdash \omega, \quad (3.10)$$

satisfying the following conditions:

- (a) For any $g, h \in G$ we have

$$\text{Jac}'(f, g) \vdash \Omega'_{f, h} \subset \Omega'_{f, gh}, \quad (3.11)$$

and the $\text{Jac}'(f, \text{id})$ -module structure on $\Omega'_{f, g}$ coincides with the $\text{Jac}(f)$ -module structure on $\Omega_{f, g}$ given by Corollary 9.

- (b) By choosing a nowhere vanishing N -form, we have the following isomorphism

$$\text{Jac}'(f, G) \xrightarrow{\cong} \Omega'_{f, G}, \quad X \mapsto X \vdash \zeta, \quad (3.12)$$

where ζ is the residue class in $\Omega'_{f, \text{id}} = \Omega_f$ of the N -form. Namely, $\Omega'_{f, G}$ is a free $\text{Jac}'(f, G)$ -module of rank one.

- (iv) There is an induce action of $\text{Aut}(f, G)$ on $\text{Jac}'(f, G)$ given by

$$\varphi^*(X) \vdash \varphi^*(\zeta) := \varphi^*(X \vdash \zeta), \quad \varphi \in \text{Aut}(f, G), \quad X \in \text{Jac}'(f, G), \quad (3.13)$$

where ζ is an element in $\Omega'_{f, \text{id}}$ giving the isomorphism in axiom (iiib). The algebra structure of $\text{Jac}'(f, G)$ satisfies the following conditions:

- (a) It is $\text{Aut}(f, G)$ -invariant, namely,

$$\varphi^*(X) \circ \varphi^*(Y) = \varphi^*(X \circ Y), \quad \varphi \in \text{Aut}(f, G), \quad X, Y \in \text{Jac}'(f, G). \quad (3.14)$$

- (b) It is G -twisted $\mathbb{Z}/2\mathbb{Z}$ -graded commutative, namely, for any $g, h \in G$ and $X \in \text{Jac}'(f, g)$, $Y \in \text{Jac}'(f, h)$, we have

$$X \circ Y = (-1)^{\bar{X} \cdot \bar{Y}} g^*(Y) \circ X, \quad (3.15)$$

where $\bar{X} = N - N_g$ and $\bar{Y} = N - N_h$ are the $\mathbb{Z}/2\mathbb{Z}$ -grading of X and Y , and g^* is the induced action of g considered as an element of $\text{Aut}(f, G)$.

- (v) For any $g, h \in G$ and $X \in \text{Jac}'(f, g)$, $\omega \in \Omega'_{f, h}$, $\omega' \in \Omega'_{f, G}$, we have

$$J_{f, G}(X \vdash \omega, \omega') = (-1)^{\bar{X} \cdot \bar{\omega}} J_{f, G}(\omega, ((h^{-1})^* X) \vdash \omega'), \quad (3.16)$$

where $\bar{X} = N - N_g$ and $\bar{\omega} = N - N_h$ are the $\mathbb{Z}/2\mathbb{Z}$ -grading of X and ω , and $(h^{-1})^*$ is the induced action of h^{-1} considered as an element of $\text{Aut}(f, G)$.

- (vi) Let G' be a finite subgroup of G_f such that $G \subseteq G'$. Fix a nowhere vanishing N -form and denote by ζ its residue class in $\Omega'_{f, \text{id}}$. By axiom (iiib) for G, G' , fix the isomorphisms given by ζ ;

$$\text{Jac}'(f, G) \xrightarrow{\cong} \Omega'_{f, G}, \quad X \mapsto X \vdash \zeta, \quad (3.17)$$

$$\text{Jac}'(f, G') \xrightarrow{\cong} \Omega'_{f, G'}, \quad X' \mapsto X' \vdash \zeta. \quad (3.18)$$

Then, the injective map $\Omega'_{f, G} \longrightarrow \Omega'_{f, G'}$ induced by the identity maps $\Omega'_{f, g} \longrightarrow \Omega'_{f, g}$, $g \in G$ yields an injective map of the $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -modules $\text{Jac}'(f, G) \rightarrow \text{Jac}'(f, G')$, which is an algebra-homomorphism.

3.3. Comments on the axioms. Such a class $\zeta \in \Omega'_{f,\text{id}}$ giving the isomorphism in axiom (iiib) is a non-zero constant multiple of the residue class of $dx_1 \wedge \cdots \wedge dx_N$. It follows that the $\text{Aut}(f, G)$ -action on $\text{Jac}'(f, G)$ does not depend on the choice of ζ . In particular, the $\text{Aut}(f, G)$ -action on $\text{Jac}'(f, \text{id}) = \text{Jac}(f)$ is nothing but the usual one which is induced by the natural $\text{Aut}(f, G)$ -action on $\mathbb{C}[x_1, \dots, x_N]$. For different choices of ζ we get isomorphic algebras.

Axioms (iva), (ivb) and (v) are naturally expected by keeping the skew group ring $\mathbb{C}[x_1, \dots, x_N] * G$ in mind (see also Remark 15). Indeed, our axioms are motivated by some intuitive properties of the ‘‘Jacobian algebra of f in the non-commutative ring $\mathbb{C}[x_1, \dots, x_N] * G$ ’’. Axiom (ivb) can also be found in [K03] under the name of ‘‘Ramond algebra for Jacobian Frobenius algebras’’, while the others seem to be new.

We have not used the commutativity of G in the axioms in Definition 16 except for the last one (vi). Instead of G_f there, by the use of the largest subgroup $G_{f,nc}$ of $\text{GL}(N; \mathbb{C})$ respecting f whose restriction f^g to $\text{Fix}(g)$ gives a finite dimensional Jacobian algebra $\text{Jac}(f^g)$ for all $g \in G_{f,nc}$, the definition can naturally be extended to the non-abelian case, namely, the case when G is a subgroup of $G_{f,nc}$.

3.4. Conjecture and the definition. We shall denote the residue class of $1 \in \mathbb{C}[x_1, \dots, x_N]$ in $\text{Jac}'(f, \text{id}) = \text{Jac}(f)$ by v_{id} ; this is the unit with respect to the product structure \circ since by axiom (v) we have

$$J_{f,G}((X \circ v_{\text{id}}) \vdash \zeta, \omega) = J_{f,G}(X \vdash (v_{\text{id}} \vdash \zeta), \omega) = J_{f,G}(X \vdash \zeta, \omega) \quad (3.19)$$

for all $X \in \text{Jac}'(f, G)$, $\omega \in \Omega'_{f,G}$ and $\zeta \in \Omega_{f,\text{id}}$ giving the isomorphism (3.12). Note also that $\varphi^*(v_{\text{id}}) = v_{\text{id}}$ for all $\varphi \in \text{Aut}(f, G)$ since $\varphi^*(v_{\text{id}}) \vdash \varphi^*(\zeta) = \varphi^*(v_{\text{id}} \vdash \zeta) = \varphi^*(\zeta) = v_{\text{id}} \vdash \varphi^*(\zeta)$. In particular, v_{id} is G -invariant.

By the isomorphism (3.12), it follows from (3.6) that

$$\varphi^*(\text{Jac}'(f, g)) = \text{Jac}'(f, \varphi \circ g \circ \varphi^{-1}), \quad \varphi \in \text{Aut}(f, G). \quad (3.20)$$

In particular, $g^*(\text{Jac}'(f, h)) = \text{Jac}'(f, ghg^{-1})$ for $g, h \in G$. Now, G is a commutative group, we have $g^*(\text{Jac}'(f, h)) = \text{Jac}'(f, h)$. Since the product structure \circ is also G -invariant by axiom (iva) it follows that the G -invariant subspace of $\text{Jac}'(f, G)$ has a structure of a $\mathbb{Z}/2\mathbb{Z}$ -graded commutative algebra, which is $\mathbb{Z}/2\mathbb{Z}$ -graded commutative due to axiom (ivb).

A priori there might not be a unique $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -algebra satisfying the axioms in Definition 16, nevertheless we expect the following

Conjecture 17. *Let the notations be as above.*

- (a) *A G -twisted Jacobian algebra $\text{Jac}'(f, G)$ of f should exist.*
- (b) *The subalgebra $(\text{Jac}'(f, G))^G$ should be uniquely determined by (f, G) up to isomorphism.*

Definition 18. Suppose that Conjecture 17 holds for the pair (f, G) . The $\mathbb{Z}/2\mathbb{Z}$ -graded commutative algebra

$$\text{Jac}(f, G) := (\text{Jac}'(f, G))^G \quad (3.21)$$

is called the *orbifold Jacobian algebra* of (f, G) .

In Theorem 22 we prove Conjecture 17 (actually a stronger statement than it), for a large class of polynomials f — so-called invertible polynomials and any symmetry group G of it.

Under the isomorphism in axiom (iiib), it follows from axiom (v) that the non-degenerate G -twisted $\mathbb{Z}/2\mathbb{Z}$ -graded symmetric \mathbb{C} -bilinear form $J_{f,G}$ on $\Omega'_{f,G}$ equips $\text{Jac}'(f, G)$ with the structure of $\mathbb{Z}/2\mathbb{Z}$ -graded Frobenius algebra. If G is a subgroup of G_f^{SL} , then $\text{age}(g) \in \mathbb{Z}$ for all $g \in G$, the residue class ζ is G -invariant and the pairing $J_{f,G}$ induces a $\mathbb{Z}/2\mathbb{Z}$ -graded symmetric pairing on $\Omega_{f,G}$ due to the G -twisted $\mathbb{Z}/2\mathbb{Z}$ -graded commutativity (Proposition 13). Therefore, it follows

easily that $\text{Jac}(f, G)$ for $G \subseteq G_f^{\text{SL}}$ is equipped with a structure of $\mathbb{Z}/2\mathbb{Z}$ -graded commutative Frobenius algebra, which will be of our main interest.

4. ORBIFOLD JACOBIAN ALGEBRAS FOR INVERTIBLE POLYNOMIALS

4.1. Invertible polynomials. A polynomial $f \in \mathbb{C}[x_1, \dots, x_N]$ is called a *weighted homogeneous* polynomial if there are positive integers w_1, \dots, w_N and d such that

$$f(\lambda^{w_1} x_1, \dots, \lambda^{w_N} x_N) = \lambda^d f(x_1, \dots, x_N) \quad (4.1)$$

for all $\lambda \in \mathbb{C}^*$. We call $(w_1, \dots, w_N; d)$ a system of weights of f . A weighted homogeneous polynomial f is called *non-degenerate* if it has at most an isolated critical point at the origin in \mathbb{C}^N , equivalently, if the Jacobian algebra $\text{Jac}(f)$ of f is finite-dimensional.

Definition 19. A weighted homogeneous polynomial $f \in \mathbb{C}[x_1, \dots, x_N]$ is called *invertible* if the following conditions are satisfied.

- (i) The number of variables ($= N$) coincides with the number of monomials in the polynomial f , namely,

$$f(x_1, \dots, x_N) = \sum_{i=1}^N c_i \prod_{j=1}^N x_j^{E_{ij}} \quad (4.2)$$

for some coefficients $c_i \in \mathbb{C}^*$ and non-negative integers E_{ij} for $i, j = 1, \dots, N$.

- (ii) The matrix $E := (E_{ij})$ is invertible over \mathbb{Q} .
 (iii) The polynomial f and the *Berglund–Hübsch transpose* \tilde{f} of f defined by

$$\tilde{f}(x_1, \dots, x_N) := \sum_{i=1}^N c_i \prod_{j=1}^N x_j^{E_{ji}} \quad (4.3)$$

are non-degenerate.

Definition 20. Let $f(x_1, \dots, x_N) = \sum_{i=1}^N c_i \prod_{j=1}^N x_j^{E_{ij}}$ be an invertible polynomial. Define rational numbers q_1, \dots, q_N by the unique solution of the equation

$$E \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \quad (4.4)$$

Namely, set $q_i := w_i/d$, $i = 1, \dots, N$, for the system of weights $(w_1, \dots, w_N; d)$.

If $f(x_1, \dots, x_N)$ is an invertible polynomial, then we have

$$G_f = \left\{ (\lambda_1, \dots, \lambda_N) \in (\mathbb{C}^*)^N \left| \prod_{j=1}^N \lambda_j^{E_{1j}} = \dots = \prod_{j=1}^N \lambda_j^{E_{Nj}} = 1 \right. \right\}, \quad (4.5)$$

and hence G_f is a finite group. It is easy to see that G_f contains an element $g_0 := (q_1, \dots, q_N)$.

It is important to note the following

Proposition 21. *The group $G_f^{\text{SL}} = G_f \cap \text{SL}(N; \mathbb{C})$ is a proper subgroup of G_f .*

Proof. Let \tilde{f} be the Berglund–Hübsch transpose of f . It is known by [ET11] and [Kr] (see also Proposition 2 in [EG-ZT]) that

$$G_f^{\text{SL}} \cong \text{Hom}(G_{\tilde{f}} / \langle (\tilde{q}_1, \dots, \tilde{q}_N) \rangle, \mathbb{C}^*) \subsetneq \text{Hom}(G_{\tilde{f}}, \mathbb{C}^*) \cong G_f,$$

where $(\tilde{q}_1, \dots, \tilde{q}_N)$ is the unique solution of the equation $(\tilde{q}_1, \dots, \tilde{q}_N)E = (1, \dots, 1)$. \square

The following is our first theorem of this paper.

Theorem 22. *Let f be an invertible polynomial and G a subgroup of G_f . There exists a unique G -twisted Jacobian algebra $\text{Jac}'(f, G)$ of f up to isomorphism. Namely, it is uniquely characterized by the axioms in Definition 16.*

In particular, the orbifold Jacobian algebra $\text{Jac}(f, G)$ of (f, G) exists.

In the subsequent subsections, we first prepare some notation, and then prove the uniqueness, and finally prove the existence.

4.2. Notations. Let $f(x_1, \dots, x_N) = \sum_{i=1}^N c_i \prod_{j=1}^N x_j^{E_{ij}}$ be an invertible polynomial. Without loss of generality one may assume that $c_i = 1$ for $i = 1, \dots, N$ by rescaling the variables. According to [KS], an invertible polynomial f can be written as a Thom–Sebastiani sum $f = f_1 \oplus \dots \oplus f_p$ of invertible polynomials (in groups of different variables) f_ν , $\nu = 1, \dots, p$ of the following types:

- (i) $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \dots + x_{m-1}^{a_{m-1}} x_m + x_m^{a_m}$ (chain type; $m \geq 1$);
- (ii) $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \dots + x_{m-1}^{a_{m-1}} x_m + x_m^{a_m} x_1$ (loop type; $m \geq 2$).

Remark 23. In [KS] the authors distinguished also polynomials of the so called Fermat type: $x_1^{a_1}$, which is regarded as a chain type polynomial with $m = 1$ in this paper.

We shall use the monomial basis of the Jacobian algebra $\text{Jac}(f_\nu)$

Proposition 24 (cf. [Kreu]). *For an invertible polynomial*

$$f_\nu = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \dots + x_{m-1}^{a_{m-1}} x_m + x_m^{a_m}$$

of chain type with $m \geq 1$, the Jacobian algebra $\text{Jac}(f_\nu)$ has a monomial basis consisting of all the monomials $x_1^{k_1} \dots x_m^{k_m}$ such that

- 1) $0 \leq k_i \leq a_i - 1$,
- 2) if

$$k_i = \begin{cases} a_i - 1 & \text{for all odd } i, i \leq 2s - 1, \\ 0 & \text{for all even } i, i \leq 2s - 1, \end{cases}$$

then $k_{2s} = 0$.

For an invertible polynomial $f_\nu = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \dots + x_{m-1}^{a_{m-1}} x_m + x_m^{a_m} x_1$ of loop type with $m \geq 2$, the Jacobian algebra $\text{Jac}(f_\nu)$ has a monomial basis consisting of all the monomials $x_1^{k_1} \dots x_m^{k_m}$ with $0 \leq k_i \leq a_i - 1$.

Let $I_g := \{i_1, \dots, i_{N_g}\}$ be a subset of $\{1, \dots, N\}$ such that $\text{Fix}(g) = \{x \in \mathbb{C}^N \mid x_j = 0, j \notin I_g\}$. In particular, $I_{\text{id}} = \{1, \dots, N\}$. Denote by I_g^c the complement of I_g in I_{id} .

In what follows, we are mostly interested in special pairs of elements of G_f .

Definition 25. Let $f = f(x_1, \dots, x_N)$ be an invertible polynomial.

- (i) An ordered pair (g, h) of elements of G_f is called *spanning* if

$$I_g \cup I_h \cup I_{gh} = \{1, \dots, N\}. \quad (4.6)$$

- (ii) For a spanning pair (g, h) of elements of G_f , define $I_{g,h} := I_g^c \cap I_h^c$.
- (iii) For a spanning pair (g, h) of elements of G_f , there always exist $g_1, g_2, h_1, h_2 \in G_f$ such that $g = g_1 g_2$ and $h = h_1 h_2$ with $g_2 h_2 = \text{id}$ and $I_{g_1, h_1} = \emptyset$. The tuple (g_1, g_2, h_1, h_2) is called the *factorization* of (g, h) .

Remark 26. For a spanning pair (g, h) of elements of G_f , up to a reordering of the variables, we have

$$\begin{aligned} g &= (0, \dots, 0, \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q) \\ h &= (\gamma_1, \dots, \gamma_r, 0, \dots, 0, 1 - \beta_1, \dots, 1 - \beta_q), \end{aligned} \quad (4.7)$$

for some rational numbers $0 < \alpha_i, \beta_i, \gamma_i < 1$ and integers p, q, r such that $0 \leq r \leq N_g$ and $N_g + p + q = r + N_h + q = N$. In this presentation, we have $I_g \cap I_h = \{i_{r+1}, \dots, i_{N-q-p}\}$, $I_{g,h} = \{i_{N-q+1}, \dots, i_N\}$ and

$$\begin{aligned} g_1 &= (0, \dots, 0, \alpha_1, \dots, \alpha_p, 0, \dots, 0), \\ g_2 &= (0, \dots, 0, 0, \dots, 0, \beta_1, \dots, \beta_q), \\ h_1 &= (\gamma_1, \dots, \gamma_r, 0, \dots, 0, 0, \dots, 0), \\ h_2 &= (0, \dots, 0, 0, \dots, 0, 1 - \beta_1, \dots, 1 - \beta_q). \end{aligned}$$

We introduce one of most important objects in this paper.

Definition 27. Let $f = f(x_1, \dots, x_N)$ be an invertible polynomial. For each spanning pair (g, h) of elements of G_f , define a polynomial $H_{g,h} \in \mathbb{C}[x_1, \dots, x_N]$ by

$$H_{g,h} := \begin{cases} \tilde{m}_{g,h} \det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j \in I_{g,h}} & \text{if } I_{g,h} \neq \emptyset \\ 1 & \text{if } I_{g,h} = \emptyset \end{cases}, \quad (4.8)$$

where $\tilde{m}_{g,h} \in \mathbb{C}^*$ is the constant uniquely determined by the following equation in $\text{Jac}(f^{g \cap h})$

$$\frac{1}{\mu_{f^{g \cap h}}} [\text{hess}(f^{g \cap h}) H_{g,h}] = \frac{1}{\mu_{f^{gh}}} [\text{hess}(f^{gh})], \quad (4.9)$$

where $f^{g \cap h}$ is an invertible polynomial given by the restriction $f|_{\text{Fix}(g) \cap \text{Fix}(h)}$ of f to the locus $\text{Fix}(g) \cap \text{Fix}(h)$.

Remark 28. The polynomial $H_{g,h}$ is a non-zero constant multiple of the determinant of a minor of the Hessian matrix of $f(x_1, \dots, x_N)$. Since $I_g \cap I_h \subseteq I_{gh}$ and $I_{g,h} \subseteq I_{gh}$, $\text{hess}(f^{g \cap h})$ and $H_{g,h}$ define elements of $\text{Jac}(f^{gh})$.

Remark 29. Let (g, h) be a spanning pair of elements of G_f . Suppose that $\text{Fix}(g) = \{\mathbf{0}\}$. Then $h = g^{-1}$. It is easy to check that $H_{g,h} = \frac{1}{\mu_f} [\text{hess}(f)]$ by the explanation of $\tilde{m}_{g,h}$ below. Recall also Definition 2 that if $\text{Fix}(g) \cap \text{Fix}(h) = \{\mathbf{0}\}$ then $\mu_{f^{g \cap h}} = 1$ and $\text{hess}(f^{g \cap h}) = 1$.

We explain the existence and the uniqueness of $\tilde{m}_{g,h}$ in Definition 27. Suppose that $f = f_1 \oplus \dots \oplus f_p$ is a Thom–Sebastiani sum such that each $f_\nu = f_\nu(x_{i_1}, \dots, x_{i_m})$, $\nu = 1, \dots, p$ is either of chain type or loop type. Set $I_\nu := \{i_1, \dots, i_m\} \subseteq \{1, \dots, N\}$ for each ν . Then $\text{Jac}(f) = \text{Jac}(f_1) \otimes \dots \otimes \text{Jac}(f_p)$ and

$$\det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j \in I_{\text{id}}} = \prod_{\nu=1}^p \det \left(\frac{\partial^2 f_\nu}{\partial x_i \partial x_j} \right)_{i,j \in I_\nu}. \quad (4.10)$$

For each $g \in G_f$ and f_ν as above the following holds:

- If f_ν is of the chain type $f_\nu = x_{i_1}^{a_1} x_{i_2} + \dots + x_{i_{m-1}}^{a_{m-1}} x_{i_m} + x_{i_m}^{a_m}$, then there exists l , $0 \leq l \leq m$ such that $\{i_1, \dots, i_l\} \subseteq I_g^c$ and $\{i_{l+1}, \dots, i_m\} \subseteq I_g$.
- If f_ν is of loop type, then $I_\nu \subseteq I_g$ or $I_\nu \subseteq I_g^c$.

We classify the possible cases of $I_{g,h}$.

Lemma 30. Let (g, h) be a spanning pair of elements of G_f . Suppose that $f = f_1 \oplus \dots \oplus f_p$ is a Thom–Sebastiani sum such that each f_ν , $\nu = 1, \dots, p$ is either of chain type or loop type. Then, for each $f_\nu = f_\nu(x_{i_1}, \dots, x_{i_m})$, the either one of the following holds:

- (i) f_ν is of chain type and, for some $0 \leq l \leq m$,
 - (a) $\{i_1, \dots, i_m\} \subseteq I_g$, $\{i_1, \dots, i_l\} \subseteq I_h^c$ and $\{i_{l+1}, \dots, i_m\} \subseteq I_h$,
 - (a') $\{i_1, \dots, i_m\} \subseteq I_h$, $\{i_1, \dots, i_l\} \subseteq I_g^c$ and $\{i_{l+1}, \dots, i_m\} \subseteq I_g$,
 - (b) $\{i_1, \dots, i_l\} \subseteq I_{g,h}$ and $\{i_{l+1}, \dots, i_m\} \subseteq I_g \cap I_h$.

- (ii) f_ν is of loop type and
- (a) $\{i_1, \dots, i_m\} \subseteq I_g \cap I_h$,
 - (b) $\{i_1, \dots, i_m\} \subseteq I_g \cap I_h^c$,
 - (b') $\{i_1, \dots, i_m\} \subseteq I_g^c \cap I_h$,
 - (c) $\{i_1, \dots, i_m\} \subseteq I_{g,h}$.

Proof. It is straightforward from the explicit form of an invertible polynomial of each type and the group action on it. \square

Obviously, only polynomials f_ν satisfying $I_\nu \cap I_{g,h} \neq \emptyset$ contribute non-trivially to $H_{g,h}$. Such an f_ν satisfies the either one of the following two by Lemma 30:

- (a) $I_\nu = \{i_1, \dots, i_m\} \subseteq I_{g,h}$.
- (b) f_ν is of the chain type and, for some $0 \leq l \leq m-1$,

$$\{i_1, \dots, i_l\} \subseteq I_{g,h} \quad \text{and} \quad \{i_{l+1}, \dots, i_m\} \subseteq I_g \cap I_h.$$

Set $\Gamma_a := \{\nu \mid f_\nu \text{ satisfies (a)}\}$ and $\Gamma_b := \{\nu \mid f_\nu \text{ satisfies (b)}\}$. Since $I_{gh} = I_{g,h} \cup (I_g \cap I_h)$, we have

$$f^{gh} = \bigoplus_{\nu_a \in \Gamma_a} f_{\nu_a} \oplus \bigoplus_{\nu_b \in \Gamma_b} f_{\nu_b} \oplus \bigoplus_{\substack{\nu \text{ s.t.} \\ I_\nu \subseteq I_g \cap I_h}} f_\nu, \quad (4.11)$$

where \oplus denotes a Thom-Sebastiani sum and hence

$$\text{Jac}(f^{gh}) = \bigotimes_{\nu_a \in \Gamma_a} \text{Jac}(f_{\nu_a}) \otimes \bigotimes_{\nu_b \in \Gamma_b} \text{Jac}(f_{\nu_b}) \otimes \bigotimes_{\substack{\nu \text{ s.t.} \\ I_\nu \subseteq I_g \cap I_h}} \text{Jac}(f_\nu). \quad (4.12)$$

Consider the factorization

$$\det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j \in I_{g,h}} = \prod_{\nu_a \in \Gamma_a} \tilde{H}_a^{(\nu_a)} \cdot \prod_{\nu_b \in \Gamma_b} \tilde{H}_b^{(\nu_b)}, \quad (4.13)$$

where

$$\tilde{H}_a^{(\nu_a)} := \det \left(\frac{\partial^2 f_{\nu_a}}{\partial x_i \partial x_j} \right)_{i,j \in I_{\nu_a}}, \quad \tilde{H}_b^{(\nu_b)} := \det \left(\frac{\partial^2 f_{\nu_b}}{\partial x_i \partial x_j} \right)_{i,j \in I_{\nu_b} \cap I_{g,h}}. \quad (4.14)$$

Suppose for simplicity that $f_{\nu_b} = x_1^{a_1} x_2 + \dots + x_{m-1}^{a_{m-1}} x_m + x_m^{a_m}$ with $I_{\nu_b} \cap I_{g,h} = \{1, \dots, l\}$. By a direct calculation, we have the following equalities in $\text{Jac}(f_{\nu_b})$;

$$[\tilde{H}_b^{(\nu_b)}] = \left(\prod_{i=1}^l a_i \right) \cdot \left(\sum_{j=1}^l (-1)^{l-j} \prod_{i=1}^j a_i \right) [x_1^{a_1-2} x_2^{a_2-1} \dots x_l^{a_l-1} x_{l+1}], \quad (4.15a)$$

$$[\text{hess}(f_{\nu_b} |_{\text{Fix}(g) \cap \text{Fix}(h)})] = \left(\prod_{i=l+1}^m a_i \right) \cdot \left(\sum_{j=l}^m (-1)^{m-j} \prod_{i=l+1}^j a_i \right) [x_{l+1}^{a_{l+1}-2} x_{l+2}^{a_{l+2}-1} \dots x_m^{a_m-1}], \quad (4.15b)$$

$$[\text{hess}(f_{\nu_b})] = \left(\prod_{i=1}^m a_i \right) \cdot \left(\sum_{j=0}^m (-1)^{m-j} \prod_{i=1}^j a_i \right) [x_1^{a_1-2} x_2^{a_2-1} \dots x_m^{a_m-1}]. \quad (4.15c)$$

Note that

$$\mu_{f_{\nu_b}} = \sum_{j=0}^m (-1)^{m-j} \prod_{i=1}^j a_i, \quad \mu_{f_{\nu_b} |_{\text{Fix}(g) \cap \text{Fix}(h)}} = \sum_{j=l}^m (-1)^{m-j} \prod_{i=l+1}^j a_i. \quad (4.16)$$

Hence, it is straightforward to see the existence and the uniqueness of $\tilde{m}_{g,h}$.

Proposition 31. *Let $f = f(x_1, \dots, x_N)$ be an invertible polynomial. For each spanning pair (g, h) of elements of G_f , the following holds:*

- (i) The class of $H_{g,h}$ is non-zero in $\text{Jac}(f^{gh})$.
- (ii) If $I_{g,h} = \emptyset$, then $[H_{g,g^{-1}H_{h,h^{-1}}}] = [H_{gh,(gh)^{-1}}]$ in $\text{Jac}(f)$.
- (iii) For any $j \in I_{g,h}$, the class of $x_j H_{g,h}$ is zero in $\text{Jac}(f^{gh})$.

Proof. Let the notations be as above. We may assume that $I_{g,h} \neq \emptyset$ since if $I_{g,h} = \emptyset$ the statements are trivially true. The part (i) is almost clear by the equation (4.9) since $[\text{hess}(f^{gh})]$ is non-zero. The part (ii) follows from the normalization of $H_{g,h}$ by the equation (4.9) in view of the equations (4.15).

To prove part (iii), first note that there is ν , $1 \leq \nu \leq p$ such that $j \in I_\nu$ for some f_ν satisfying either one of (a) or (b) above. Due to the factorization of $\text{Jac}(f^{gh})$, it is enough to show that $[x_j \tilde{H}_a^{(\nu)}] = 0$ if $\nu \in \Gamma_a$ and $[x_j \tilde{H}_b^{(\nu)}] = 0$ if $\nu \in \Gamma_b$. Since the first case is almost clear, suppose that $f_\nu \in \Gamma_b$, $I_\nu = \{1, \dots, m\}$ and $I_\nu \cap I_{g,h} = \{1, \dots, l\}$. Recall again that $[\tilde{H}_b^{(\nu)}]$ is a non-zero constant multiple of $[x_1^{a_1-2} x_2^{a_2-1} \dots x_l^{a_l-1} x_{l+1}]$. It is easy to calculate by induction that $[x_1^{a_1-1} x_2] = 0$ and $[x_j^{a_j} x_{j+1}] = 0$ in $\text{Jac}(f_\nu)$ for $j = 2, \dots, l$. Therefore, we have $[x_j \tilde{H}_b^{(\nu)}] = 0$ in $\text{Jac}(f_\nu)$ for $j = 1, \dots, l$ (see also the description of the monomial basis in Proposition 24). This completes the part (iii) of the proposition. \square

Proposition 32. *Let $f = f(x_1, \dots, x_N)$ be an invertible polynomial. For each spanning pair (g, h) of elements of G_f , we have*

$$(N - N_g) + (N - N_h) \equiv (N - N_{gh}) \pmod{2}. \quad (4.17)$$

Moreover, if $I_{g,h} = \emptyset$ then $(N - N_g) + (N - N_h) = (N - N_{gh})$.

Proof. First of all, note that $N - N_g = |I_g^c|$ — the number of elements in the set I_g^c . Therefore, the following equalities yield the statement:

$$\begin{aligned} N - N_g &= |I_g^c \setminus I_{g,h}| + |I_{g,h}|, & N - N_h &= |I_h^c \setminus I_{g,h}| + |I_{g,h}|, \\ N - N_{gh} &= |I_{gh}^c| = |I_g^c \setminus I_{g,h}| + |I_h^c \setminus I_{g,h}|. \end{aligned}$$

\square

For each $g \in G_f$, the set $I_g \subseteq \{1, \dots, N\}$ and its complement I_g^c will often be regarded as a subsequence of $(1, \dots, N)$:

$$I_g = (i_1, \dots, i_{N_g}), \quad i_1 < \dots < i_{N_g}, \quad I_g^c = (j_1, \dots, j_{N-N_g}), \quad j_1 < \dots < j_{N-N_g}. \quad (4.18)$$

Definition 33. Let g_1, \dots, g_k be elements of G_f such that $I_{g_i, g_j} = \emptyset$ if $i \neq j$.

- (i) Denote by $I_{g_1}^c \sqcup I_{g_2}^c$ the sequence given by adding the sequence $I_{g_2}^c$ at the end of the sequence $I_{g_1}^c$. Define inductively $I_{g_1}^c \sqcup \dots \sqcup I_{g_k}^c$ by $\left(I_{g_1}^c \sqcup \dots \sqcup I_{g_{k-1}}^c \right) \sqcup I_{g_k}^c$. Obviously, as a set, $I_{g_1}^c \sqcup \dots \sqcup I_{g_k}^c = I_{g_1 \dots g_k}^c$.
- (ii) Let σ_{g_1, \dots, g_k} be the permutation which turns the sequence $I_{g_1}^c \sqcup \dots \sqcup I_{g_k}^c$ to the sequence $I_{g_1 \dots g_k}^c$. Define $\tilde{\varepsilon}_{g_1, \dots, g_k}$ as the signature $\text{sgn}(\sigma_{g_1, \dots, g_k})$ of the permutation σ_{g_1, \dots, g_k} .

It is straightforward from the definition that

$$\tilde{\varepsilon}_{g, \text{id}} = 1 = \tilde{\varepsilon}_{\text{id}, g}, \quad g \in G_f, \quad (4.19a)$$

$$\tilde{\varepsilon}_{g, h} = (-1)^{(N-N_g)(N-N_h)} \tilde{\varepsilon}_{h, g}, \quad g, h \in G_f, \quad I_{g, h} = \emptyset, \quad (4.19b)$$

$$\tilde{\varepsilon}_{g, g'} \tilde{\varepsilon}_{g g', g''} = \tilde{\varepsilon}_{g, g'} \tilde{\varepsilon}_{g', g''} = \tilde{\varepsilon}_{g, g' g''} \tilde{\varepsilon}_{g', g''}, \quad g, g', g'' \in G_f, \quad I_{g, g'} = I_{g', g''} = I_{g, g''} = \emptyset. \quad (4.19c)$$

4.3. Uniqueness. Throughout this subsection, $f = f(x_1, \dots, x_N)$ denotes an invertible polynomial. In this subsection, we shall show that for any $G \subseteq G_f$ the axioms in Definition 16 determine uniquely $\text{Jac}'(f, G)$ up to isomorphism. We only have to show that for $g, h \in G$ the product $\circ : \text{Jac}'(f, g) \otimes_{\mathbb{C}} \text{Jac}'(f, h) \rightarrow \text{Jac}'(f, gh)$ is uniquely determined up to rescaling of generators of $\text{Jac}(f^g)$ -modules $\text{Jac}'(f, g)$.

Take a nowhere vanishing N -form $dx_1 \wedge \cdots \wedge dx_N$ and set $\zeta := [dx_1 \wedge \cdots \wedge dx_N] \in \Omega_f$. For each subgroup $G \subseteq G_f$, fix an isomorphism in axiom (iii)

$$\vdash: \text{Jac}'(f, G) \xrightarrow{\cong} \Omega'_{f, G}, \quad X \mapsto X \vdash \zeta, \quad (4.20)$$

where ζ is considered as an element in $\Omega'_{f, \text{id}} = \Omega_f$ (recall Definition 10). Fix also a map

$$\alpha: G_f \longrightarrow \mathbb{C}^*, \quad g \mapsto \alpha_g, \quad (4.21)$$

such that $\alpha_{\text{id}} = 1$ and

$$\alpha_g \alpha_{g^{-1}} = (-1)^{\frac{1}{2}(N-N_g)(N-N_g+1)}, \quad g \in G_f. \quad (4.22)$$

Such a map α always exists since for each g we may choose α_g as

$$\alpha_g = \mathbf{e} \left[\frac{1}{8}(N-N_g)(N-N_g+1) \right]. \quad (4.23)$$

For each $g \in G$, let v_g be an element of $\text{Jac}'(f, g)$, such that

$$v_g \vdash \zeta = \alpha_g \omega_g, \quad (4.24)$$

where $\omega_g \in \Omega'_{f, g}$ is the residue class of $\tilde{\omega}_g \in \Omega^{N_g}(\text{Fix}(g))$ and

$$\tilde{\omega}_g := \begin{cases} dx_{i_1} \wedge \cdots \wedge dx_{i_{N_g}} & \text{if } I_g = (i_1, \dots, i_{N_g}), i_1 < \cdots < i_{N_g} \\ 1_g & \text{if } I_g = \emptyset \end{cases}. \quad (4.25)$$

Obviously, we have $\omega_{\text{id}} = \zeta$.

Remark 34. It might not be necessary to distinguish ζ and ω_{id} , however, we regard ζ as a “primitive form” (cf. [S1, S2, ST]) at the origin of the base space of the “properly-defined deformation space” of the pair (f, G) while we hold ω_{id} as just a $\text{Jac}'(f, \text{id})$ -basis of $\Omega'_{f, \text{id}}$.

By axiom (i), we have $\text{Jac}'(f, \text{id}) = \text{Jac}(f)$. Therefore, $v_{\text{id}} = [1]$ and, by axioms (ii) and (iiia), $v_{\text{id}} \circ v_g = v_g \circ v_{\text{id}} = v_g$. Axiom (iiia) implies that for all $Y \in \text{Jac}'(f, g)$ there exists $X \in \text{Jac}'(f, \text{id}) = \text{Jac}(f)$ represented by a polynomial in $\{x_i\}_{i \in I_g}$ such that $Y = X \circ v_g$. For any $X \in \text{Jac}'(f, \text{id})$, we shall often write $X \circ v_g$ as $X|_{\text{Fix}(g)} v_g$ where $X|_{\text{Fix}(g)}$ is the image of X under the map $\text{Jac}(f) \longrightarrow \text{Jac}(f^g)$.

Proposition 35. *For a pair (g, h) of elements of G which is not spanning, we have*

$$v_g \circ v_h = 0 \in \text{Jac}'(f, G).$$

Proof. Denote by $[\gamma'_{g, h}(\mathbf{x})]$ the element of $\text{Jac}(f^{gh})$ satisfying $v_g \circ v_h = [\gamma'_{g, h}(\mathbf{x})] v_{gh}$. Suppose that $f = f_1 \oplus \cdots \oplus f_p$ is a Thom–Sebastiani sum such that each f_ν , $\nu = 1, \dots, p$ is either of chain type or loop type. Without loss of generality, we may assume the coordinate x_k , $k \notin I_g \cup I_h \cup I_{gh}$ to be a variable of the polynomial f_1 . Consider the following two cases;

- (a) $f_1 = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{m-1}^{a_{m-1}} x_m + x_m^{a_m}$ is of chain type.
- (b) $f_1 = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{m-1}^{a_{m-1}} x_m + x_m^{a_m} x_1$ is of loop type.

Case (a): First, note that $1 \notin I_g \cup I_h \cup I_{gh}$. Consider $(\frac{1}{a_1}, 0, \dots, 0) \in \text{Aut}(f_1, G)$ and extend it naturally to the element $\varphi \in \text{Aut}(f, G)$. Since $1 \notin I_g \cup I_h \cup I_{gh}$, we have $\varphi^*(v_{g'}) = \mathbf{e} \left[-\frac{1}{a_1} \right] v_{g'}$ for $g' \in \{g, h, gh\}$. Axiom (iva) yields $\varphi^*([\gamma'_{g, h}(\mathbf{x})]) = \mathbf{e} \left[-\frac{1}{a_1} \right] [\gamma'_{g, h}(\mathbf{x})]$. On the other hand, we have $\varphi^*([\gamma'_{g, h}(\mathbf{x})]) = [\gamma'_{g, h}(\mathbf{x})]$ since $1 \notin I_{gh}$. Hence, $[\gamma'_{g, h}(\mathbf{x})] = 0$.

Case (b): First, note that $1, \dots, m \notin I_g \cup I_h \cup I_{gh}$. Choose an element of $G_{f_1} \setminus G_{f_1}^{\text{SL}}$, which exists due to Proposition 21, and extend it naturally to the element $\varphi \in \text{Aut}(f, G)$. There exists a complex number $\lambda_\varphi \neq 1$, the determinant of φ regarded as an element of $\text{GL}(N; \mathbb{C})$, such that $\varphi^*(v_{g'}) = \lambda_\varphi^{-1} v_{g'}$ for $g' \in \{g, h, gh\}$ since $1, \dots, m \notin I_g \cup I_h \cup I_{gh}$. Axiom iva

yields $\varphi^*([\gamma'_{g,h}(\mathbf{x})]) = \lambda_\varphi^{-1}[\gamma'_{g,h}(\mathbf{x})]$. On the other hand, we have $\varphi^*([\gamma'_{g,h}(\mathbf{x})]) = [\gamma'_{g,h}(\mathbf{x})]$ since $1, \dots, m \notin I_{gh}$. Hence, $[\gamma'_{g,h}(\mathbf{x})] = 0$. \square

We consider the product $v_g \circ v_h$ for a spanning pair (g, h) .

Proposition 36. *For each spanning pair (g, h) of elements of G , there exists $c_{g,h} \in \mathbb{C}$ such that*

$$v_g \circ v_h = c_{g,h}[H_{g,h}]v_{gh}. \quad (4.26)$$

Moreover, $c_{g,h}$ does not depend on the choice of the subgroup G of G_f containing g, h .

Proof. We only need to show the first statement since the second one follows from it together with axiom (vi), the definition of v_g in (4.24) and the independence of $H_{g,h}$ from a particular choice of G . Based on Lemma 30, we study which variable in f_ν can appear in the product structure.

Lemma 37. *Let the notation and the cases be as in Lemma 30 above. There is a polynomial $\gamma_{g,h}(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_N]$ which doesn't depend on x_{i_1}, \dots, x_{i_m} such that the one of the following holds:*

- (i) (a) $v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}$.
- (b) $v_g \circ v_h = \begin{cases} \left[\gamma_{g,h}(\mathbf{x}) \cdot \left(x_{i_1}^{a_{i_1}-2} x_{i_2}^{a_{i_2}-1} \dots x_{i_m}^{a_{i_m}-1} \right) \right] v_{gh} & \text{if } l = m \\ \left[\gamma_{g,h}(\mathbf{x}) \cdot \left(x_{i_1}^{a_{i_1}-2} x_{i_2}^{a_{i_2}-1} \dots x_{i_l}^{a_{i_l}-1} x_{i_{l+1}} \right) \right] v_{gh} & \text{if } l < m \end{cases}$,
- (ii) (a) $v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}$.
- (b) $v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}$.
- (c) $v_g \circ v_h = \left[\gamma_{g,h}(\mathbf{x}) \cdot \left(x_{i_1}^{a_{i_1}-1} x_{i_2}^{a_{i_2}-1} \dots x_{i_m}^{a_{i_m}-1} \right) \right] v_{gh}$.

Here, we denote by $[\gamma_{g,h}(\mathbf{x})]$ the class of $\gamma_{g,h}(\mathbf{x})$ in $\text{Jac}(f^{gh})$.

Proof. (i): We may assume $f_\nu = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \dots + x_m^{a_m}$. For each $r = 1, \dots, m$, there is a unique element $\varphi_r \in \text{Aut}(f_\nu, G)$ such that $\varphi_r(x_i) = x_i$ for all $i = r+1, \dots, m$, which is explicitly given by

$$\begin{aligned} \varphi_r(x_r) &:= \mathbf{e} \begin{bmatrix} 1 \\ a_r \end{bmatrix} x_r, \\ \varphi_r(x_i) &:= \mathbf{e} \left[\frac{1}{a_i} \left(1 - \frac{1}{a_{i+1}} \left(1 - \dots - \frac{1}{a_{r-1}} \left(1 - \frac{1}{a_r} \right) \right) \right) \right] x_i, \quad 1 \leq i < r. \end{aligned}$$

Denote also by φ_p its natural extension to $\text{Aut}(f, G)$ and by $\lambda_{\varphi_r} \in \mathbb{C}^*$ the determinant of φ_r regarded as an element of $\text{GL}(N; \mathbb{C})$.

- (a) For each $r = 1, \dots, m$, we have $\varphi_r^*(v_g) = v_g$, $\varphi_r^*(v_h) = \lambda_{\varphi_r}^{-1} v_h$ and $\varphi_r^*(v_{gh}) = \lambda_{\varphi_r}^{-1} v_{gh}$. Suppose that a polynomial $\gamma_{g,h}(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_N]$ satisfies $v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}$. By axiom (iva), we obtain

$$\begin{aligned} [\varphi_r^*(\gamma_{g,h}(\mathbf{x}))]v_{gh} &= \lambda_{\varphi_r} \varphi_r^*([\gamma_{g,h}(\mathbf{x})]v_{gh}) = \lambda_{\varphi_r} \varphi_r^*(v_g \circ v_h) \\ &= \lambda_{\varphi_r} \varphi_r^*(v_g) \circ \varphi_r^*(v_h) = v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}, \end{aligned}$$

and hence $\varphi_r^*([\gamma_{g,h}(\mathbf{x})]) = [\gamma_{g,h}(\mathbf{x})]$ in $\text{Jac}(f^{gh})$. In view of the above action of φ_r and Proposition 24, the polynomial $\gamma_{g,h}(\mathbf{x})$ can be chosen so that it does not depend on x_i , $i = 1, \dots, m$.

- (b) For each $r = 1, \dots, m$, we have $\varphi_r^*(v_g) = \lambda_{\varphi_r}^{-1} v_g$, $\varphi_r^*(v_h) = \lambda_{\varphi_r}^{-1} v_h$ and $\varphi_r^*(v_{gh}) = v_{gh}$. Suppose that a polynomial $\gamma'_{g,h}(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_N]$ satisfies $v_g \circ v_h = [\gamma'_{g,h}(\mathbf{x})]v_{gh}$. By axiom (iva), we obtain

$$\begin{aligned} [\varphi_r^*(\gamma'_{g,h}(\mathbf{x}))]v_{gh} &= \varphi_r^*([\gamma'_{g,h}(\mathbf{x})]v_{gh}) = \varphi_r^*(v_g \circ v_h) \\ &= \varphi_r^*(v_g) \circ \varphi_r^*(v_h) = \lambda_{\varphi_r}^{-2} (v_g \circ v_h) = \lambda_{\varphi_r}^{-2} [\gamma'_{g,h}(\mathbf{x})]v_{gh}, \end{aligned}$$

and hence $[\varphi_r^*(\gamma'_{g,h}(\mathbf{x}))] = \lambda_{\varphi_r}^{-2}[\gamma'_{g,h}(\mathbf{x})]$ in $\text{Jac}(f^{gh})$. In view of the above action of φ_r and Proposition 24, the polynomial $\gamma'_{g,h}(\mathbf{x})$ can be chosen so that it is divisible by $x_1^{a_1-2}x_2^{a_2-1}\dots x_m^{a_m-1}$ if $l = m$ and by $x_1^{a_1-2}x_2^{a_2-1}\dots x_l^{a_l-1}x_{l+1}$ if $l < m$.

(ii): We may assume $f_\nu = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \dots + x_m^{a_m}x_1$. For each element $\varphi \in G_{f_\nu}$ regarded as an element of $\text{Aut}(f_\nu, G)$, denote also by φ its natural extension to $\text{Aut}(f, G)$. Let $\lambda_\varphi \in \mathbb{C}^*$ be the determinant of φ regarded as an element of $\text{GL}(N; \mathbb{C})$. Note that if $\varphi \neq \text{id}$ then $\varphi(x_i) \neq x_i$ for all $i = 1, \dots, m$.

- (a) For all $\varphi \in G_{f_\nu}$, we have $\varphi^*(v_g) = v_g$, $\varphi^*(v_h) = v_h$ and $\varphi^*(v_{gh}) = v_{gh}$. Suppose that a polynomial $\gamma_{g,h}(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_N]$ satisfies $v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}$. By axiom (iva), we obtain

$$\begin{aligned} [\varphi^*(\gamma_{g,h}(\mathbf{x}))]v_{gh} &= \varphi^*(\gamma_{g,h}(\mathbf{x})v_{gh}) = \varphi^*(v_g \circ v_h) \\ &= \varphi^*(v_g) \circ \varphi^*(v_h) = v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}, \end{aligned}$$

and hence $[\varphi^*(\gamma_{g,h}(\mathbf{x}))] = [\gamma_{g,h}(\mathbf{x})]$ in $\text{Jac}(f^{gh})$. In view of Proposition 24, the polynomial $\gamma_{g,h}(\mathbf{x})$ can be chosen so that it does not depend on x_i , $i = 1, \dots, m$.

- (b) Suppose that a polynomial $\gamma_{g,h}(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_N]$ satisfies $v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}$. Since $1, \dots, m$ do not belong to $I_g \cap I_h$ nor $I_{g,h}$, it is obvious that the polynomial $\gamma_{g,h}(\mathbf{x})$ can be chosen so that it does not depend on x_i , $i = 1, \dots, m$.
- (c) For all $\varphi \in G_{f_\nu}$, we have $\varphi^*(v_g) = \lambda_\varphi^{-1}v_g$, $\varphi^*(v_h) = \lambda_\varphi^{-1}v_h$ and $\varphi^*(v_{gh}) = v_{gh}$. Suppose that a polynomial $\gamma'_{g,h}(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_N]$ satisfies $v_g \circ v_h = [\gamma'_{g,h}(\mathbf{x})]v_{gh}$. By axiom (iva), we obtain

$$\begin{aligned} [\varphi^*(\gamma'_{g,h}(\mathbf{x}))]v_{gh} &= \varphi^*(\gamma'_{g,h}(\mathbf{x})v_{gh}) = \varphi^*(v_g \circ v_h) \\ &= \varphi^*(v_g) \circ \varphi^*(v_h) = \lambda_\varphi^{-2}(v_g \circ v_h) = \lambda_\varphi^{-2}[\gamma'_{g,h}(\mathbf{x})]v_{gh}, \end{aligned}$$

and hence $[\varphi^*(\gamma'_{g,h}(\mathbf{x}))] = \lambda_\varphi^{-2}[\gamma'_{g,h}(\mathbf{x})]$ in $\text{Jac}(f^{gh})$. In view of Proposition 24, the polynomial $\gamma'_{g,h}(\mathbf{x})$ can be chosen so that it is divisible by $x_1^{a_1-1}x_2^{a_2-1}\dots x_m^{a_m-1}$. □

Now the first statement of the proposition is a direct consequence of Lemma 37, since $H_{g,h}$ is a constant multiple of the product of the monomials in the round brackets there. We have finished the proof of the proposition. □

By Proposition 36, we may assume that $G = G_f$. We give some properties of $c_{g,h}$.

Lemma 38. *For each $g \in G_f$, we have*

$$c_{g,g^{-1}} = (-1)^{\frac{1}{2}(N-N_g)(N-N_g-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g) \right]. \quad (4.27)$$

Proof. We have

$$\begin{aligned} \frac{1}{\mu_{fg}} J_{f,g}([\text{hess}(f^g)]v_g \vdash \zeta, v_{g^{-1}} \vdash \zeta) &= \frac{\alpha_g \alpha_{g^{-1}}}{\mu_{fg}} J_{f,g}([\text{hess}(f^g)]\omega_g, \omega_{g^{-1}}) \\ &= (-1)^{\frac{1}{2}(N-N_g)(N-N_g-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g) \right] \cdot |G|. \end{aligned}$$

On the other hand, by axiom (v) and normalization (4.9) of $H_{g,h}$, we have

$$\begin{aligned}
\frac{1}{\mu_{f^g}} J_{f,g}([\text{hess}(f^g)]v_g \vdash \zeta, v_{g^{-1}} \vdash \zeta) &= \frac{1}{\mu_{f^g}} J_{f,id}(\omega_{id}, [\text{hess}(f^g)]v_g \circ v_{g^{-1}} \vdash \zeta) \\
&= \frac{1}{\mu_{f^g}} J_{f,id}(\omega_{id}, c_{g,g^{-1}}[\text{hess}(f^g)H_{g,g^{-1}}]\omega_{id}) \\
&= \frac{c_{g,g^{-1}}}{\mu_f} J_{f,id}(\omega_{id}, [\text{hess}(f)]\omega_{id}) \\
&= c_{g,g^{-1}}|G|.
\end{aligned}$$

□

Lemma 39. *For each pair (g, h) of elements of G_f such that $I_{g,h} = \emptyset$, we have*

$$c_{g,h}c_{h^{-1},g^{-1}} = (-1)^{(N-N_g)(N-N_h)}. \quad (4.28)$$

In particular it follows that $c_{g,h} \neq 0$.

Remark 40. If $I_{g,h} = \emptyset$ for a pair (g, h) of elements of G_f , it is spanning.

Proof. We have

$$\begin{aligned}
&v_g \circ (v_h \circ v_{h^{-1}}) \circ v_{g^{-1}} \\
&= (-1)^{\frac{1}{2}(N-N_g)(N-N_g-1) + \frac{1}{2}(N-N_h)(N-N_h-1)} \cdot \mathbf{e} \left[-\frac{1}{2}\text{age}(g) - \frac{1}{2}\text{age}(h) \right] [g^*(H_{h,h^{-1}})H_{g,g^{-1}}]v_{id}, \\
&(v_g \circ v_h) \circ (v_{h^{-1}} \circ v_{g^{-1}}) \\
&= (-1)^{\frac{1}{2}(N-N_{gh})(N-N_{gh}-1)} \mathbf{e} \left[-\frac{1}{2}\text{age}(gh) \right] c_{g,h}c_{h^{-1},g^{-1}}[H_{gh,(gh)^{-1}}]v_{id}.
\end{aligned}$$

The proposition follows from the facts that the product \circ is associative, $g^*(H_{h,h^{-1}}) = H_{h,h^{-1}}$ since $I_{g,h} = \emptyset$, $[H_{g,g^{-1}}H_{h,h^{-1}}] = [H_{gh,(gh)^{-1}}]$ in $\text{Jac}(f)$, $\text{age}(g) + \text{age}(h) = \text{age}(gh)$ since $I_{g,h} = \emptyset$, and $(N - N_g) + (N - N_h) \equiv (N - N_{gh}) \pmod{2}$ by Proposition 32. □

Corollary 41. *Let (g, h) be a spanning pair of elements of G_f with the factorization (g_1, g_2, h_1, h_2) . The complex numbers c_{g_1,h_2} , c_{g_2,h_1} and c_{g_1,h_1} are non-zero.*

Proof. It follows from the fact that $I_{g_1,h_2} = \emptyset$, $I_{g_2,h_1} = \emptyset$ and $I_{g_1,h_1} = \emptyset$. □

Proposition 42. *Let (g, h) be a spanning pair of elements of G_f with the factorization (g_1, g_2, h_1, h_2) . We have*

$$c_{g,h} = (-1)^{\frac{1}{2}(N-N_{g_2})(N-N_{g_2}-1)} \cdot \mathbf{e} \left[-\frac{1}{2}\text{age}(g_2) \right] \cdot \frac{c_{g_1,h_1}}{c_{g_1,g_2}c_{h_2,h_1}}. \quad (4.29)$$

In particular, $c_{g,h} \neq 0$.

Proof. We have

$$\begin{aligned}
v_{g_1} \circ (v_{g_2} \circ v_{h_2}) \circ v_{h_1} &= (-1)^{\frac{1}{2}(N-N_{g_2})(N-N_{g_2}-1)} \cdot \mathbf{e} \left[-\frac{1}{2}\text{age}(g_2) \right] \cdot v_{g_1} \circ [H_{g_2,g_2^{-1}}]v_{id} \circ v_{h_1} \\
&= (-1)^{\frac{1}{2}(N-N_{g_2})(N-N_{g_2}-1)} \cdot \mathbf{e} \left[-\frac{1}{2}\text{age}(g_2) \right] \cdot c_{g_1,h_1}[H_{g_2,g_2^{-1}}]v_{gh}.
\end{aligned}$$

On the other hand, we get:

$$\begin{aligned}
(v_{g_1} \circ v_{g_2}) \circ (v_{h_2} \circ v_{h_1}) &= c_{g_1,g_2}e_{g_1,g_2} \circ c_{h_2,h_1}v_{h_1h_2} \\
&= c_{g_1,g_2}c_{h_2,h_1}c_{g,h}[H_{g,h}]v_{gh}.
\end{aligned}$$

Note that $H_{g,h} = H_{g_2,g_2^{-1}} = H_{h_2,h_2^{-1}}$ by the definition of the factorization (g_1, g_2, h_1, h_2) . By Corollary 41, we know that c_{g_1,g_2} and c_{h_2,h_1} are non-zero, which gives the statement. □

Hence, by this proposition, we only have to determine $c_{g,h}$ for all pairs (g, h) of elements of G_f such that $I_{g,h} = \emptyset$.

Suppose that $f = f_1 \oplus \cdots \oplus f_p$ is a Thom–Sebastiani sum such that each f_ν , $\nu = 1, \dots, p$ is either of chain type or loop type. Then, we have a natural isomorphism $G_f \cong G_{f_1} \times \cdots \times G_{f_p}$. Therefore, it follows that each $g \in G_f$ has the unique expression $g = g_1 \cdots g_p$ such that $g_i \in G_{f_i}$ for all $i = 1, \dots, p$, hence $I_{g_i, g_j} = \emptyset$ if $i \neq j$ and $I_g^c = I_{g_1}^c \cup \cdots \cup I_{g_p}^c$. Under this notation, define \tilde{v}_g by

$$\tilde{v}_g := \tilde{\varepsilon}_{g_1, \dots, g_p} v_{g_1} \circ \cdots \circ v_{g_p}. \quad (4.30)$$

Obviously, \tilde{v}_g is a non-zero constant multiple of v_g for all $g \in G_f$. It is also easy to see that \tilde{v}_g does not depend on the choice of ordering in the Thom–Sebastiani sum and that for a pair (g, h) of elements of G_f with $I_{g,h} = \emptyset$ we have

$$\tilde{v}_g \circ \tilde{v}_h = \frac{1}{\tilde{\varepsilon}_{g,h}} \tilde{v}_{gh}. \quad (4.31)$$

Proposition 43. *For each $g \in G$, we have*

$$\tilde{v}_g \circ \tilde{v}_{g^{-1}} = (-1)^{\frac{1}{2}(N-N_g)(N-N_g-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g) \right] \cdot [H_{g,g^{-1}}] \tilde{v}_{\text{id}}. \quad (4.32)$$

Proof. There is an inductive presentation of \tilde{v}_g given by

$$\tilde{v}_g = \begin{cases} v_{g_1} & \text{if } g = g_1, \\ \tilde{\varepsilon}_{g_1 \dots g_i, g_{i+1}} \tilde{v}_{g_1 \dots g_i} \circ v_{g_{i+1}} & \text{if } g = g_1 \dots g_i g_{i+1}, \quad i = 1, \dots, p-1. \end{cases}$$

The statement follows by induction from the following calculation:

$$\begin{aligned} \tilde{v}_g \circ \tilde{v}_{g^{-1}} &= (\tilde{\varepsilon}_{g_1 \dots g_i, g_{i+1}} \tilde{v}_{g_1 \dots g_i} \circ v_{g_{i+1}}) \circ (\tilde{\varepsilon}_{g_1^{-1} \dots g_i^{-1}, g_{i+1}^{-1}} \tilde{v}_{g_1^{-1} \dots g_i^{-1}} \circ v_{g_{i+1}^{-1}}) \\ &= (-1)^{(N-N_{g_1^{-1} \dots g_i^{-1}})(N-N_{g_{i+1}})} \cdot (\tilde{v}_{g_1 \dots g_i} \circ \tilde{v}_{g_1^{-1} \dots g_i^{-1}}) \circ (v_{g_{i+1}} \circ v_{g_{i+1}^{-1}}) \\ &= (-1)^{(N-N_{g_1 \dots g_i})(N-N_{g_{i+1}}) + \frac{1}{2}(N-N_{g_1 \dots g_i})(N-N_{g_1 \dots g_i} - 1) + \frac{1}{2}(N-N_{g_{i+1}})(N-N_{g_{i+1}} - 1)} \\ &\quad \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_1 \dots g_i) - \frac{1}{2} \text{age}(g_{i+1}) \right] \cdot [H_{g_1 \dots g_i, g_1^{-1} \dots g_i^{-1}} H_{g_{i+1}, g_{i+1}^{-1}}] \tilde{v}_{\text{id}} \\ &= (-1)^{\frac{1}{2}(N-N_g)(N-N_g-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g) \right] \cdot [H_{g,g^{-1}}] \tilde{v}_{\text{id}}. \end{aligned}$$

□

This proposition says that by replacing the map $\alpha : G_f \rightarrow \mathbb{C}^*$ to the suitable one we have a new basis $\{\tilde{v}_g\}_{g \in G_f}$ instead of $\{v_g\}_{g \in G_f}$. To summarize, we finally obtain the following

Corollary 44. *Let (g, h) be a spanning pair of elements of G_f with the factorization (g_1, g_2, h_1, h_2) . We have*

$$\tilde{v}_g \circ \tilde{v}_h = (-1)^{\frac{1}{2}(N-N_{g_2})(N-N_{g_2}-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_2) \right] \cdot \frac{\tilde{\varepsilon}_{g_1, g_2} \tilde{\varepsilon}_{h_2, h_1}}{\tilde{\varepsilon}_{g_1, h_1}} [H_{g,h}] \tilde{v}_{gh}. \quad (4.33)$$

In particular, for any subgroup G of G_f , if a G -twisted Jacobian algebra of f exists, then it is uniquely determined by the axioms in Definition 16 up to isomorphism.

4.4. Existence. In this subsection, we prove the existence of a G -twisted Jacobian algebra of f . We first show this when $G = G_f$.

Definition 45. Define a $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -module $\mathcal{A}' = \mathcal{A}'_0 \oplus \mathcal{A}'_1$ as follows: for each $g \in G_f$, consider a free $\text{Jac}(f^g)$ -module \mathcal{A}'_g of rank one generated by a formal letter \bar{v}_g ,

$$\mathcal{A}'_g = \text{Jac}(f^g) \bar{v}_g. \quad (4.34a)$$

and set

$$\mathcal{A}'_0 := \bigoplus_{\substack{g \in G_f \\ N - N_g \equiv 0 \pmod{2}}} \mathcal{A}'_g, \quad \text{and} \quad \mathcal{A}'_1 := \bigoplus_{\substack{g \in G_f \\ N - N_g \equiv 1 \pmod{2}}} \mathcal{A}'_g. \quad (4.34b)$$

By definition, axiom (i) trivially holds for \mathcal{A}' .

Definition 46. For a spanning pair (g, h) of elements of G_f with the factorization (g_1, g_2, h_1, h_2) , set

$$\bar{c}_{g,h} := (-1)^{\frac{1}{2}(N - N_{g_2})(N - N_{g_2} - 1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_2) \right] \cdot \frac{\tilde{\varepsilon}_{g_1, g_2} \tilde{\varepsilon}_{h_2, h_1}}{\tilde{\varepsilon}_{g_1, h_1}}. \quad (4.35)$$

It is also easy to see that

$$\bar{c}_{g, \text{id}} = 1 = \bar{c}_{\text{id}, g}, \quad g \in G_f, \quad (4.36a)$$

$$\bar{c}_{g, g^{-1}} = (-1)^{\frac{1}{2}(N - N_g)(N - N_g - 1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g) \right], \quad g \in G_f, \quad (4.36b)$$

$$\bar{c}_{g,h} = \tilde{\varepsilon}_{g,h}^{-1}, \quad g, h \in G_f, \quad I_{g,h} = \emptyset. \quad (4.36c)$$

Definition 47. For each $g, h \in G_f$, define an element of \mathcal{A}'_{gh} by

$$\bar{v}_g \circ \bar{v}_h := \begin{cases} \bar{c}_{g,h} [H_{g,h}] \bar{v}_{gh} & \text{if the pair } (g, h) \text{ is spanning} \\ 0 & \text{otherwise} \end{cases}. \quad (4.37)$$

It is clear that $\bar{v}_{\text{id}} \circ \bar{v}_g = \bar{v}_g = \bar{v}_g \circ \bar{v}_{\text{id}}$ since $I_{\text{id},g} = I_{g,\text{id}} = \emptyset$ and hence $[H_{\text{id},g}] = [H_{g,\text{id}}] = 1$.

Proposition 48. For a spanning pair (g, h) of elements of G_f which has the factorization (g_1, g_2, h_1, h_2) , we have

$$\bar{c}_{g,h} = (-1)^{(N - N_g)(N - N_h)} \cdot \mathbf{e} [-\text{age}(g_2)] \cdot \bar{c}_{h,g}. \quad (4.38)$$

Hence, we have

$$\bar{v}_g \circ \bar{v}_h = (-1)^{(N - N_g)(N - N_h)} \cdot (\mathbf{e} [-\text{age}(g_2)] \bar{v}_h \circ \bar{v}_g). \quad (4.39)$$

Proof. We have

$$\begin{aligned} \bar{c}_{g,h} &= (-1)^{\frac{1}{2}(N - N_{g_2})(N - N_{g_2} - 1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_2) \right] \cdot \frac{\tilde{\varepsilon}_{g_1, g_2} \tilde{\varepsilon}_{h_2, h_1}}{\tilde{\varepsilon}_{g_1, h_1}} \\ &= (-1)^{(N - N_{g_1})(N - N_{g_2}) + (N - N_{h_1})(N - N_{h_2}) - (N - N_{g_1})(N - N_{h_1}) + (N - N_{g_2})} \cdot \mathbf{e} [-\text{age}(g_2)] \\ &\quad \cdot (-1)^{\frac{1}{2}(N - N_{h_2})(N - N_{h_2} - 1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(h_2) \right] \cdot \frac{\tilde{\varepsilon}_{h_1, h_2} \tilde{\varepsilon}_{g_2, g_1}}{\tilde{\varepsilon}_{h_1, g_1}} \\ &= (-1)^{(N - N_g)(N - N_h)} \cdot \mathbf{e} [-\text{age}(g_2)] \cdot \bar{c}_{h,g}, \end{aligned}$$

where we used that $h_2 = g_2^{-1}$, $N - N_{g_2} = \text{age}(g_2) + \text{age}(h_2)$ and Proposition 32. \square

Proposition 49. For each $g, g', g'' \in G_f$, we have

$$(\bar{v}_g \circ \bar{v}_{g'}) \circ \bar{v}_{g''} = \bar{v}_g \circ (\bar{v}_{g'} \circ \bar{v}_{g''}). \quad (4.40)$$

Proof. First, we show the following

Lemma 50. For $g, g', g'' \in G_f$, suppose that (g, g') and $(g g', g'')$ are spanning pairs with $I_{g,g'} \subseteq I_{g''}$.

(i) There exist $g_1, g_2, g_3, g'_1, g'_2, g'_3, g''_1, g''_2, g''_3 \in G_f$ such that

$$g = g_1 g_2 g_3, \quad g' = g'_1 g'_2 g'_3, \quad g'' = g''_1 g''_2 g''_3, \quad g'_1 g'_1 = \text{id}, \quad g_2 g'_2 = \text{id}, \quad g_3 g'_3 = \text{id}, \quad (4.41)$$

and $(g_1 g_2, g_3, g'_1 g'_2, g'_3)$ is the factorization of (g, g') and $(g_1 g'_2, g_2 g'_1, g''_3, g'_1 g''_2)$ is the factorization of $(g g', g'')$.

(ii) The pairs (g', g'') and $(g, g' g'')$ are spanning such that $I_{g', g''} \subseteq I_g$.

Proof. (i) Similarly to the presentation of (4.7), the elements g, g', g'' satisfying the conditions can be expressed, in the multiplicative form, as follows:

$$\begin{aligned} g &= g_1 \cdot g_2 \cdot \text{id} \cdot \text{id} \cdot g_3 \cdot \text{id} \\ g' &= \text{id} \cdot \text{id} \cdot g'_1 \cdot g'_2 \cdot g'_3 \cdot \text{id}. \\ g'' &= \text{id} \cdot g''_2 \cdot g''_1 \cdot \text{id} \cdot \text{id} \cdot g''_3 \end{aligned} \quad (4.42)$$

(ii) By the above presentation, it is easy to see that (g, g') and (gg', g'') are spanning pairs. It follows from $g'_1 g''_1 = \text{id}$ that $I_{g', g''} \subseteq I_g$. \square

Lemma 51. *The LHS of (4.40) is non-zero if and only if the RHS of (4.40) is non-zero.*

Proof. By Proposition 31 (iii), the LHS of (4.40) is non-zero only if both pairs (g, g') and (gg', g'') are spanning and $I_{g, g'} \subseteq I_{g''}$ and the RHS of (4.40) is non-zero only if both pairs $(g, g'g'')$ and (g', g'') are spanning and $I_{g', g''} \subseteq I_g$. Lemma 50 together with Proposition 48 yields the statement. \square

Lemma 52. *Let the notations be as above. We have*

$$H_{g, g'} = H_{g_3, g'_3}, \quad H_{gg', g''} = H_{g_2 g'_1, g''_2 g'_1}, \quad H_{g, g'g''} = H_{g_2 g_3, g'_2 g'_3}, \quad H_{g', g''} = H_{g'_1, g''_1}, \quad (4.43)$$

and hence $[H_{g, g'} H_{gg', g''}] = [H_{g, g'g''} H_{g', g''}]$ in $\text{Jac}(f^{gg'g''})$.

Proof. The first statement follows from the definition of $H_{g, h}$ and the second one does from Proposition 31 (ii). \square

Therefore, we only have to show the following

Lemma 53. *Let the notations be as above. we have*

$$\bar{c}_{g, g'} \bar{c}_{gg', g''} = \bar{c}_{g, g'g''} \bar{c}_{g', g''}. \quad (4.44)$$

Proof. It follows from the definition (4.35) that

$$\begin{aligned} \bar{c}_{g, g'} &= (-1)^{\frac{1}{2}(N-N_{g_3})(N-N_{g_3}-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_3) \right] \cdot \frac{\tilde{\varepsilon}_{g_1 g_2, g_3} \tilde{\varepsilon}_{g'_3, g'_1 g'_2}}{\tilde{\varepsilon}_{g_1 g_2, g'_1 g'_2}}, \\ \bar{c}_{gg', g''} &= (-1)^{\frac{1}{2}(N-N_{g_2 g'_1})(N-N_{g_2 g'_1}-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_2 g'_1) \right] \cdot \frac{\tilde{\varepsilon}_{g_1 g'_2, g_2 g'_1} \tilde{\varepsilon}_{g'_2 g'_1, g'_3}}{\tilde{\varepsilon}_{g_1 g'_2, g'_3}}, \\ \bar{c}_{g, g'g''} &= (-1)^{\frac{1}{2}(N-N_{g_2 g_3})(N-N_{g_2 g_3}-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_2 g_3) \right] \cdot \frac{\tilde{\varepsilon}_{g_1, g_2 g_3} \tilde{\varepsilon}_{g'_2 g'_3, g'_2 g'_3}}{\tilde{\varepsilon}_{g_1, g'_2 g'_3}}, \\ \bar{c}_{g', g''} &= (-1)^{\frac{1}{2}(N-N_{g'_1})(N-N_{g'_1}-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g'_1) \right] \cdot \frac{\tilde{\varepsilon}_{g'_2 g'_3, g'_1} \tilde{\varepsilon}_{g''_1, g''_2 g''_3}}{\tilde{\varepsilon}_{g'_2 g'_3, g''_2 g''_3}}. \end{aligned}$$

Since all $I_{g_i}^c$, $I_{g'_i}^c$ and $I_{g''_i}^c$ are mutually disjoint, we get

$$\begin{aligned}
\bar{c}_{g,g'} \bar{c}_{gg',g''} &= (-1)^{\frac{1}{2}(N-N_{g_3})(N-N_{g_3}-1) + \frac{1}{2}(N-N_{g_2g'_1})(N-N_{g_2g'_1}-1)} \\
&\quad \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_3) - \frac{1}{2} \text{age}(g_2g'_1) \right] \cdot \frac{\tilde{\varepsilon}_{g_1g_2g_3} \tilde{\varepsilon}_{g'_3g'_1g'_2} \tilde{\varepsilon}_{g_1g'_2g_2g'_1} \tilde{\varepsilon}_{g'_2g'_1g'_3}}{\tilde{\varepsilon}_{g_1g_2g'_1g'_2} \tilde{\varepsilon}_{g_1g'_2g'_3}} \\
&= (-1)^{\frac{1}{2}(N-N_{g_3})(N-N_{g_3}-1) + \frac{1}{2}(N-N_{g_2}+N-N_{g'_1})(N-N_{g_2}+N-N_{g'_1}-1)} \\
&\quad \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_3) - \frac{1}{2} \text{age}(g_2) - \frac{1}{2} \text{age}(g'_1) \right] \\
&\quad \cdot \frac{\tilde{\varepsilon}_{g_1g_2} \tilde{\varepsilon}_{g_1g_2g_3} \tilde{\varepsilon}_{g'_3g'_1g'_2} \tilde{\varepsilon}_{g'_1g'_2} \tilde{\varepsilon}_{g_1g'_2} \tilde{\varepsilon}_{g_1g'_2g'_1g_2} \tilde{\varepsilon}_{g'_1g_2} \tilde{\varepsilon}_{g'_1g'_2} \tilde{\varepsilon}_{g'_1g'_2g'_3}}{\tilde{\varepsilon}_{g_1g_2} \tilde{\varepsilon}_{g_1g_2g'_1g'_2} \tilde{\varepsilon}_{g'_1g'_2} \tilde{\varepsilon}_{g_1g'_2} \tilde{\varepsilon}_{g_1g'_2g'_3}} \\
&= (-1)^{\frac{1}{2} \left((N-N_{g_3})^2 - (N-N_{g_3}) + (N-N_{g_2})^2 - (N-N_{g_2}) + (N-N_{g'_1})^2 - (N-N_{g'_1}) + 2(N-N_{g_2})(N-N_{g'_1}) \right)} \\
&\quad \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_3) - \frac{1}{2} \text{age}(g_2) - \frac{1}{2} \text{age}(g'_1) \right] \\
&\quad \cdot \frac{\tilde{\varepsilon}_{g_1g_2g_3} \tilde{\varepsilon}_{g'_3g'_1g'_2} \tilde{\varepsilon}_{g_1g'_2g'_1g_2} \tilde{\varepsilon}_{g'_1g_2} \tilde{\varepsilon}_{g'_1g'_2} \tilde{\varepsilon}_{g'_1g'_2g'_3}}{\tilde{\varepsilon}_{g_1g_2g'_1g'_2} \tilde{\varepsilon}_{g_1g'_2g'_3}},
\end{aligned}$$

and

$$\begin{aligned}
\bar{c}_{g,g'} \bar{c}_{g',g''} &= (-1)^{\frac{1}{2}(N-N_{g_2g_3})(N-N_{g_2g_3}-1) + \frac{1}{2}(N-N_{g'_1})(N-N_{g'_1}-1)} \\
&\quad \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_2g_3) - \frac{1}{2} \text{age}(g'_1) \right] \cdot \frac{\tilde{\varepsilon}_{g_1g_2g_3} \tilde{\varepsilon}_{g'_2g'_3g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_1} \tilde{\varepsilon}_{g'_1g'_2g'_3}}{\tilde{\varepsilon}_{g_1g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_2g'_3}} \\
&= (-1)^{\frac{1}{2}(N-N_{g_2}+N-N_{g_3})(N-N_{g_2}+N-N_{g_3}-1) + \frac{1}{2}(N-N_{g'_1})(N-N_{g'_1}-1)} \\
&\quad \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_3) - \frac{1}{2} \text{age}(g_2) - \frac{1}{2} \text{age}(g'_1) \right] \\
&\quad \cdot \frac{\tilde{\varepsilon}_{g_1g_2g_3} \tilde{\varepsilon}_{g_2g_3} \tilde{\varepsilon}_{g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_1} \tilde{\varepsilon}_{g'_1g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_2g'_3}}{\tilde{\varepsilon}_{g_1g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_1} \tilde{\varepsilon}_{g'_1g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_2g'_3}} \\
&= (-1)^{\frac{1}{2} \left((N-N_{g_3})^2 - (N-N_{g_3}) + (N-N_{g_2})^2 - (N-N_{g_2}) + (N-N_{g'_1})^2 - (N-N_{g'_1}) + 2(N-N_{g_2})(N-N_{g_3}) \right)} \\
&\quad \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_3) - \frac{1}{2} \text{age}(g_2) - \frac{1}{2} \text{age}(g'_1) \right] \\
&\quad \cdot \frac{\tilde{\varepsilon}_{g_1g_2g_3} \tilde{\varepsilon}_{g_2g_3} \tilde{\varepsilon}_{g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_1} \tilde{\varepsilon}_{g'_1g'_2g'_3}}{\tilde{\varepsilon}_{g_1g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_2g'_3}}.
\end{aligned}$$

Therefore, we only have to show that

$$\begin{aligned}
&(-1)^{(N-N_{g_2})(N-N_{g'_1})} \cdot \frac{\tilde{\varepsilon}_{g_1g_2g_3} \tilde{\varepsilon}_{g'_3g'_1g'_2} \tilde{\varepsilon}_{g_1g'_2g'_1g_2} \tilde{\varepsilon}_{g'_1g_2} \tilde{\varepsilon}_{g'_1g'_2} \tilde{\varepsilon}_{g'_1g'_2g'_3}}{\tilde{\varepsilon}_{g_1g_2g'_1g'_2} \tilde{\varepsilon}_{g_1g'_2g'_3}} \\
&= (-1)^{(N-N_{g_2})(N-N_{g_3})} \cdot \frac{\tilde{\varepsilon}_{g_1g_2g_3} \tilde{\varepsilon}_{g_2g_3} \tilde{\varepsilon}_{g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_1} \tilde{\varepsilon}_{g'_1g'_2g'_3}}{\tilde{\varepsilon}_{g_1g'_2g'_3} \tilde{\varepsilon}_{g'_2g'_3g'_2g'_3}}.
\end{aligned}$$

Since $g'_1g''_1 = \text{id}$, $g_2g''_2 = \text{id}$ and $g_3g'_3 = \text{id}$, we have $I_{g'_1}^c = I_{g''_1}^c$, $I_{g_2}^c = I_{g''_2}^c$ and $I_{g_3}^c = I_{g'_3}^c$. We also have that $\tilde{\varepsilon}_{\bullet}^2 = 1$ for any expression \bullet . Hence, the problem is reduced to show the following

equation:

$$\begin{aligned} & (-1)^{(N-N_{g_2})(N-N_{g'_1})} \cdot \frac{\tilde{\varepsilon}_{g_1,g_2,g_3} \tilde{\varepsilon}_{g_3,g'_1,g'_2} \tilde{\varepsilon}_{g_1,g'_2,g'_1,g_2} \tilde{\varepsilon}_{g'_1,g_2,g'_3}}{\tilde{\varepsilon}_{g_1,g_2,g'_1,g'_2} \tilde{\varepsilon}_{g_1,g'_2,g'_3}} \\ &= (-1)^{(N-N_{g_2})(N-N_{g_3})} \cdot \frac{\tilde{\varepsilon}_{g_1,g_2,g_3} \tilde{\varepsilon}_{g_2,g_3,g'_2,g'_3} \tilde{\varepsilon}_{g'_2,g_3,g'_1} \tilde{\varepsilon}_{g'_1,g_2,g'_3}}{\tilde{\varepsilon}_{g_1,g'_2,g'_3} \tilde{\varepsilon}_{g'_2,g_3,g_2,g'_3}}. \end{aligned}$$

Recall also that $\tilde{\varepsilon}_\bullet$ is the signature of a permutation σ_\bullet based on the expression \bullet (see Definition 33), and hence we get a suitable sign by interchanging two indexes, for example,

$$\tilde{\varepsilon}_{g_3,g'_1,g'_2} = (-1)^{(N-N_{g'_1})(N-N_{g_2})} \tilde{\varepsilon}_{g_3,g'_2,g'_1}.$$

The LHS of the above equation is given by

$$\begin{aligned} & (-1)^{(N-N_{g_2})(N-N_{g'_1})} \cdot \frac{\tilde{\varepsilon}_{g_1,g_2,g_3} \tilde{\varepsilon}_{g_3,g'_1,g'_2} \tilde{\varepsilon}_{g_1,g'_2,g'_1,g_2} \tilde{\varepsilon}_{g'_1,g_2,g'_3}}{\tilde{\varepsilon}_{g_1,g_2,g'_1,g'_2} \tilde{\varepsilon}_{g_1,g'_2,g'_3}} \\ &= (-1)^{(N-N_{g_2})(N-N_{g'_1})} \cdot \frac{\tilde{\varepsilon}_{g_1,g_2,g_3} (-1)^{(N-N_{g'_1})(N-N_{g_2})} \tilde{\varepsilon}_{g_3,g'_2,g'_1} \tilde{\varepsilon}_{g'_1,g_2,g'_3}}{\tilde{\varepsilon}_{g_1,g'_2,g'_3}} \\ & \quad \cdot \frac{(-1)^{(N-N_{g_2})(N-N_{g'_1})+(N-N_{g_2})(N-N_{g'_2})+(N-N_{g'_2})(N-N_{g'_1})} \tilde{\varepsilon}_{g_1,g_2,g'_1,g'_2}}{\tilde{\varepsilon}_{g_1,g_2,g'_1,g'_2}} \\ &= (-1)^{(N-N_{g_2})(N-N_{g'_2})} \cdot \frac{\tilde{\varepsilon}_{g_1,g_2,g_3} \tilde{\varepsilon}_{g_3,g'_2,g'_1} \tilde{\varepsilon}_{g'_1,g_2,g'_3}}{\tilde{\varepsilon}_{g_1,g'_2,g'_3}}, \end{aligned}$$

while the RHS is given by

$$\begin{aligned} & (-1)^{(N-N_{g_2})(N-N_{g_3})} \cdot \frac{\tilde{\varepsilon}_{g_1,g_2,g_3} \tilde{\varepsilon}_{g_2,g_3,g'_2,g'_3} \tilde{\varepsilon}_{g'_2,g_3,g'_1} \tilde{\varepsilon}_{g'_1,g_2,g'_3}}{\tilde{\varepsilon}_{g_1,g'_2,g'_3} \tilde{\varepsilon}_{g'_2,g_3,g_2,g'_3}} \\ &= (-1)^{(N-N_{g_2})(N-N_{g_3})} \cdot \frac{\tilde{\varepsilon}_{g_1,g_2,g_3} (-1)^{(N-N_{g'_2})(N-N_{g_3})} \tilde{\varepsilon}_{g_3,g'_2,g'_1} \tilde{\varepsilon}_{g'_1,g_2,g'_3}}{\tilde{\varepsilon}_{g_1,g'_2,g'_3}} \\ & \quad \cdot \frac{(-1)^{(N-N_{g_2})(N-N_{g_3})+(N-N_{g_2})(N-N_{g'_2})+(N-N_{g'_2})(N-N_{g_3})} \tilde{\varepsilon}_{g'_2,g_3,g_2,g'_3}}{\tilde{\varepsilon}_{g'_2,g_3,g_2,g'_3}} \\ &= (-1)^{(N-N_{g_2})(N-N_{g'_2})} \cdot \frac{\tilde{\varepsilon}_{g_1,g_2,g_3} \tilde{\varepsilon}_{g_3,g'_2,g'_1} \tilde{\varepsilon}_{g'_1,g_2,g'_3}}{\tilde{\varepsilon}_{g_1,g'_2,g'_3}}, \end{aligned}$$

which coincides with the LHS. □

We have finished the proof of the proposition. □

Now, it is possible to equip \mathcal{A}' with a structure of $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -algebra.

Definition 54. Define a \mathbb{C} -bilinear map $\circ : \mathcal{A}' \otimes_{\mathbb{C}} \mathcal{A}' \rightarrow \mathcal{A}'$ by setting, for each $g, h \in G_f$ and $\phi(\mathbf{x}), \psi(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_N]$,

$$([\phi(\mathbf{x})]\bar{v}_g) \circ ([\psi(\mathbf{x})]\bar{v}_h) := \bar{c}_{g,h} [\phi(\mathbf{x})\psi(\mathbf{x})H_{g,h}] \bar{v}_{gh}. \quad (4.45)$$

It is easy to see that the map \circ is well-defined by Proposition 31 (iii).

Proposition 55. *The map \circ equips \mathcal{A}' with a structure of $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -algebra with the identity \bar{v}_{id} , which satisfies axiom (ii).*

Proof. The associativity of the product follows from Proposition 49. It is obvious by Proposition 32 that $\mathcal{A}'_i \circ \mathcal{A}'_j \subset \mathcal{A}'_{i+j}$ for all $i, j \in \mathbb{Z}/2\mathbb{Z}$. It is also clear by the definition of the map \circ above that the natural surjective maps $\text{Jac}(f) \rightarrow \text{Jac}(f^g)$, $g \in G_f$ equip \mathcal{A}' with a structure of $\text{Jac}(f)$ -module, which coincides with the product map $\circ : \mathcal{A}'_{\text{id}} \otimes_{\mathbb{C}} \mathcal{A}'_g \rightarrow \mathcal{A}'_g$. □

Take a nowhere vanishing N -form $dx_1 \wedge \cdots \wedge dx_N$ and set $\zeta := [dx_1 \wedge \cdots \wedge dx_N] \in \Omega_f$. For each $g \in G_f$, let $\omega_g \in \Omega'_{f,g}$ be the residue class of $\tilde{\omega}_g \in \Omega^{N_g}(\text{Fix}(g))$ where

$$\tilde{\omega}_g := \begin{cases} dx_{i_1} \wedge \cdots \wedge dx_{i_{N_g}} & \text{if } I_g = (i_1, \dots, i_{N_g}), \ i_1 < \cdots < i_{N_g} \\ 1_g & \text{if } I_g = \emptyset \end{cases}. \quad (4.46)$$

Obviously, we have $\omega_{\text{id}} = \zeta$.

Definition 56. Define a \mathbb{C} -bilinear map $\vdash: \mathcal{A}' \otimes_{\mathbb{C}} \Omega'_{f,G_f} \longrightarrow \Omega'_{f,G_f}$ by setting, for each $g, h \in G_f$ and $\phi(\mathbf{x}), \psi(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_N]$,

$$([\phi(\mathbf{x})]\bar{v}_g) \vdash ([\psi(\mathbf{x})]\omega_h) := \frac{\bar{\alpha}_{gh}\bar{c}_{g,h}}{\bar{\alpha}_h} [\phi(\mathbf{x})\psi(\mathbf{x})H_{g,h}]\omega_{gh}, \quad (4.47)$$

where $\bar{\alpha}: G \longrightarrow \mathbb{C}^*$, $g \mapsto \bar{\alpha}_g$ is a map given by

$$\bar{\alpha}_g := \mathbf{e} \left[\frac{1}{8}(N - N_g)(N - N_g + 1) \right]. \quad (4.48)$$

Remark 57. The map $\bar{\alpha}: G \longrightarrow \mathbb{C}^*$ satisfies $\bar{\alpha}_{\text{id}} = 1$ and

$$\bar{\alpha}_g \bar{\alpha}_{g^{-1}} = (-1)^{\frac{1}{2}(N - N_g)(N - N_g + 1)}, \quad g \in G_f. \quad (4.49)$$

The map \vdash induces an isomorphism $\vdash \zeta: \mathcal{A}' \longrightarrow \Omega'_{f,G_f}$ of $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -modules:

$$\vdash \zeta: \mathcal{A}'_g \longrightarrow \Omega'_{f,g}, \quad [\phi(\mathbf{x})]\bar{v}_g \mapsto [\phi(\mathbf{x})]\bar{v}_g \vdash \zeta = \bar{\alpha}_g [\phi(\mathbf{x})]\omega_g, \quad (4.50)$$

Note that for each $g, h \in G_f$ and $\phi(\mathbf{x}), \psi(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_N]$ we have

$$([\phi(\mathbf{x})]\bar{v}_g) \vdash ([\psi(\mathbf{x})]\omega_h) = (([\phi(\mathbf{x})]\bar{v}_g) \circ ([\psi(\mathbf{x})]\bar{v}_h)) \vdash \zeta, \quad (4.51)$$

by which we obtain the following

Proposition 58. *The map $\vdash: \mathcal{A}' \otimes_{\mathbb{C}} \Omega'_{f,G_f} \longrightarrow \Omega'_{f,G_f}$ satisfies axiom (iii) in Definition 16.*

On \mathcal{A}' we have the action of $\varphi \in \text{Aut}(f, G)$ induced by the isomorphism $\vdash \zeta: \mathcal{A}' \longrightarrow \Omega'_{f,G_f}$, which is denoted by φ^* . We also use the notation of (3.6).

Proposition 59. *Axiom (iv) is satisfied by \mathcal{A}' , namely, axioms (iva) and (ivb) hold.*

Proof. Let (g, h) be a spanning pair of elements of G_f with the factorization (g_1, g_2, h_1, h_2) and φ an element of $\text{Aut}(f, G)$. For simplicity, set $g' := \varphi \circ g \circ \varphi^{-1}$, $h' := \varphi \circ h \circ \varphi^{-1}$, $g'_i := \varphi \circ g_i \circ \varphi^{-1}$ and $h'_i := \varphi \circ h_i \circ \varphi^{-1}$ for $i = 1, 2$. Note that the pair (g', h') is a spanning pair with the factorization (g'_1, g'_2, h'_1, h'_2) since φ is a \mathbb{C} -algebra automorphism of $\mathbb{C}[x_1, \dots, x_N]$, which induces a bi-regular map $\varphi: (\text{Fix}(g'_i)) \longrightarrow \text{Fix}(g_i)$. It also follows that there exist $\lambda_\varphi, \lambda_{\varphi_{g_i}}, \lambda_{\varphi_{h_i}} \in \mathbb{C}^*$, $i = 1, 2$ such that

$$\varphi^*(\tilde{\omega}_{\text{id}}) = \lambda_\varphi \tilde{\omega}_{\text{id}}, \quad \varphi^*(\tilde{\omega}_{g_i}) = \lambda_{\varphi_{g_i}} \tilde{\omega}_{g'_i}, \quad \varphi^*(\tilde{\omega}_{h_i}) = \lambda_{\varphi_{h_i}} \tilde{\omega}_{h'_i}, \quad i = 1, 2,$$

(see (4.46) for the definition of $\tilde{\omega}_g$) and that, by (4.48), $\bar{\alpha}_{g'} = \bar{\alpha}_g$, $\bar{\alpha}_{h'} = \bar{\alpha}_h$, $\bar{\alpha}_{g'_i} = \bar{\alpha}_{g_i}$ and $\bar{\alpha}_{h'_i} = \bar{\alpha}_{h_i}$ for $i = 1, 2$.

For each $\phi(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_N]$, we have

$$\varphi^*([\phi(\mathbf{x})]\bar{v}_g) = [\varphi^*\phi(\mathbf{x})]\varphi^*(\bar{v}_g),$$

since

$$\begin{aligned} \varphi^*([\phi(\mathbf{x})]\bar{v}_g) \vdash \varphi^*(\zeta) &= \varphi^*([\phi(\mathbf{x})]\bar{v}_g \vdash \zeta) = \varphi^*(\bar{\alpha}_g [\phi(\mathbf{x})]\omega_g) \\ &= \bar{\alpha}_g [\varphi^*\phi(\mathbf{x})]\varphi^*(\omega_g) = \frac{\bar{\alpha}_g}{\bar{\alpha}_{g'}} ([\varphi^*\phi(\mathbf{x})]\varphi^*(\bar{v}_g)) \vdash \varphi^*(\zeta) = ([\varphi^*\phi(\mathbf{x})]\varphi^*(\bar{v}_g)) \vdash \varphi^*(\zeta). \end{aligned}$$

Therefore, we only need to show that $\varphi^*(\bar{v}_g) \circ \varphi^*(\bar{v}_h) = \varphi^*(\bar{v}_g \circ \bar{v}_h)$.

It easily follows that

$$\varphi^*(\bar{v}_{\text{id}}) = \bar{v}_{\text{id}}, \quad \varphi^*(\bar{v}_{g_i}) = \frac{\lambda_{\varphi_{g_i}}}{\lambda_\varphi} \bar{v}_{g'_i}, \quad \varphi^*(\bar{v}_{h_i}) = \frac{\lambda_{\varphi_{h_i}}}{\lambda_\varphi} \bar{v}_{h'_i}, \quad i = 1, 2,$$

since $\varphi^*(\bar{v}_{\text{id}}) \vdash \varphi^*(\zeta) = \varphi^*(\bar{v}_{\text{id}} \vdash \zeta) = \varphi^*(\zeta)$ and

$$\begin{aligned} (\lambda_{\varphi_{g_i}} \bar{v}_{g'_i}) \vdash \zeta &= \lambda_{\varphi_{g_i}} \bar{\alpha}_{g'_i} \omega_{g'_i} = \varphi^*(\bar{\alpha}_{g_i} \omega_{g_i}) = \varphi^*(\bar{v}_{g_i}) \vdash \varphi^*(\zeta) = (\lambda_\varphi \varphi^*(\bar{v}_{g_i})) \vdash \zeta, \\ (\lambda_{\varphi_{h_i}} \bar{v}_{h'_i}) \vdash \zeta &= \lambda_{\varphi_{h_i}} \bar{\alpha}_{h'_i} \omega_{h'_i} = \varphi^*(\bar{\alpha}_{h_i} \omega_{h_i}) = \varphi^*(\bar{v}_{h_i}) \vdash \varphi^*(\zeta) = (\lambda_\varphi \varphi^*(\bar{v}_{h_i})) \vdash \zeta. \end{aligned}$$

Lemma 60. *We have*

$$\varphi^*(\omega_g) = \frac{\lambda_{\varphi_{g_1}} \lambda_{\varphi_{g_2}}}{\lambda_\varphi} \cdot \frac{\tilde{\varepsilon}_{g'_1, g'_2}}{\tilde{\varepsilon}_{g'_1, g'_2}} \cdot \omega_{g'}, \quad \varphi^*(\omega_h) = \frac{\lambda_{\varphi_{h_1}} \lambda_{\varphi_{h_2}}}{\lambda_\varphi} \cdot \frac{\tilde{\varepsilon}_{h_1, h_2}}{\tilde{\varepsilon}_{h'_1, h'_2}} \cdot \omega_{h'}, \quad (4.52)$$

which implies

$$\varphi^*(\bar{v}_g) = \frac{\lambda_{\varphi_{g_1}} \lambda_{\varphi_{g_2}}}{\lambda_\varphi^2} \cdot \frac{\tilde{\varepsilon}_{g'_1, g'_2}}{\tilde{\varepsilon}_{g_1, g_2}} \cdot \bar{v}_{g'}, \quad \varphi^*(\bar{v}_h) = \frac{\lambda_{\varphi_{h_1}} \lambda_{\varphi_{h_2}}}{\lambda_\varphi^2} \cdot \frac{\tilde{\varepsilon}_{h'_1, h'_2}}{\tilde{\varepsilon}_{h_1, h_2}} \cdot \bar{v}_{h'}. \quad (4.53)$$

Proof. Let $\mathcal{T}_{\mathbb{C}^N}$ be the tangent sheaf on \mathbb{C}^N . For each $g'' \in G_f$, define a poly-vector field $\tilde{\theta}_{g''} \in \Gamma(\mathbb{C}^N, \wedge^{N-N_{g''}} \mathcal{T}_{\mathbb{C}^N})$ by

$$\tilde{\theta}_{g''} := \begin{cases} \frac{\partial}{\partial x_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{j_{N-N_{g''}}}} & \text{if } I_{g''}^c = (j_1, \dots, j_{N-N_{g''}}), \quad j_1 < \cdots < j_{N-N_{g''}}, \\ 1 & \text{if } I_{g''}^c = \emptyset \end{cases}.$$

Since we have $\varphi^*(\tilde{\omega}_{\text{id}}) = \lambda_\varphi \tilde{\omega}_{\text{id}}$ and $\varphi^*(\tilde{\omega}_{g_i}) = \lambda_{\varphi_{g_i}} \tilde{\omega}_{g'_i}$ for $i = 1, 2$, the poly-vector field $\tilde{\theta}_{g_i}$ transforms under φ as

$$\tilde{\theta}_{g_i} \mapsto \frac{\lambda_{\varphi_{g_i}}}{\lambda_\varphi} \cdot \frac{\tilde{\varepsilon}_{g_i}}{\tilde{\varepsilon}_{g'_i}} \cdot \tilde{\theta}_{g'_i}, \quad i = 1, 2,$$

where $\tilde{\varepsilon}_{g_i}$ is the signature of the permutation $I_{\text{id}} \rightarrow I_{g_i}^c \sqcup I_{g_i}$ and $\tilde{\varepsilon}_{g'_i}$ is the signature of the permutation $I_{\text{id}} \rightarrow I_{g'_i}^c \sqcup I_{g'_i}$. Suppose that $\varphi^*(\omega_g) = \lambda_{\varphi_g} \omega_{g'}$ for some $\lambda_{\varphi_g} \in \mathbb{C}^*$ and let $\tilde{\varepsilon}_g$ be the signature of the permutation $I_{\text{id}} \rightarrow I_g^c \sqcup I_g$ and $\tilde{\varepsilon}_{g'}$ be signature of the permutation $I_{\text{id}} \rightarrow I_{g'}^c \sqcup I_{g'}$. Then, $\tilde{\theta}_g$ transforms under φ as

$$\tilde{\theta}_g \mapsto \frac{\lambda_{\varphi_g}}{\lambda_\varphi} \cdot \frac{\tilde{\varepsilon}_g}{\tilde{\varepsilon}_{g'}} \cdot \tilde{\theta}_{g'},$$

Note that $\tilde{\theta}_g = \tilde{\varepsilon}_{g_1, g_2} \tilde{\theta}_{g_1} \wedge \tilde{\theta}_{g_2}$ and $\tilde{\theta}_{g'} = \tilde{\varepsilon}_{g'_1, g'_2} \tilde{\theta}_{g'_1} \wedge \tilde{\theta}_{g'_2}$. Hence, we have

$$\frac{\lambda_{\varphi_g}}{\lambda_\varphi} \cdot \frac{\tilde{\varepsilon}_g \tilde{\varepsilon}_{g'_1, g'_2}}{\tilde{\varepsilon}_{g'} \tilde{\varepsilon}_{g_1, g_2}} = \frac{\lambda_{\varphi_{g_1}} \lambda_{\varphi_{g_2}}}{\lambda_\varphi^2} \cdot \frac{\tilde{\varepsilon}_{g_1} \tilde{\varepsilon}_{g_2}}{\tilde{\varepsilon}_{g'_1} \tilde{\varepsilon}_{g'_2}}.$$

Therefore, the statement is reduced to show that

$$\frac{\tilde{\varepsilon}_{g_1} \tilde{\varepsilon}_{g_2}}{\tilde{\varepsilon}_g} = \frac{\tilde{\varepsilon}_{g'_1} \tilde{\varepsilon}_{g'_2}}{\tilde{\varepsilon}_{g'}}.$$

However, by calculating the number of elements less than j in the sequences $I_{g_1}^c$, $I_{g_2}^c$ and I_g^c for each element j in $I_{g_1}^c$ or $I_{g_2}^c$, it turns out that the LHS of the above equation is equal to $(-1)^{(N-N_{g_1})(N-N_{g_2})}$. Similarly, the RHS is equal to $(-1)^{(N-N_{g'_1})(N-N_{g'_2})}$. They coincide since we have $N_{g_1} = N_{g'_1}$ and $N_{g_2} = N_{g'_2}$. \square

Lemma 61. *We have*

$$[\varphi^* H_{g, h}] = \frac{\lambda_{\varphi_{g_2}}^2}{\lambda_\varphi^2} [H_{g', h'}]. \quad (4.54)$$

Proof. Recall Definition 27, where $H_{g,h}$ is defined as a non-zero constant multiple of

$$\det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j \in I_{g,h}}.$$

Now, $I_{g,h} = I_{g_2}^c = I_{h_2}^c$, $I_{g',h'} = I_{g_2'}^c = I_{h_2'}^c$. This is nothing but the transformation rule of the determinant under the automorphism φ . \square

Since $g_2 h_2 = \text{id}$ and $g_2' h_2' = \text{id}$ by definition of the factorizations,

$$N_{g_2} = N_{h_2} = N_{h_2'} = N_{g_2'}, \quad \lambda_{\varphi_{g_2}} = \lambda_{\varphi_{h_2}},$$

where we identify ω_{h_2} with ω_{g_2} under $\Omega_{f,h_2} = \Omega_{f,g_2}$ and $\omega_{h_2'}$ with $\omega_{g_2'}$ under $\Omega_{f,h_2'} = \Omega_{f,g_2'}$. By the above lemma, it follows that

$$\begin{aligned} & \varphi^*(\bar{v}_g) \circ \varphi^*(\bar{v}_h) \\ &= \frac{\lambda_{\varphi_{g_1}} \lambda_{\varphi_{g_2}} \lambda_{\varphi_{h_1}} \lambda_{\varphi_{h_2}}}{\lambda_{\varphi}^4} \cdot \frac{\tilde{\varepsilon}_{g_1, g_2}}{\tilde{\varepsilon}_{g_1', g_2'}} \cdot \frac{\tilde{\varepsilon}_{h_1, h_2}}{\tilde{\varepsilon}_{h_1', h_2'}} \cdot \bar{v}_{g'} \circ \bar{v}_{h'} \\ &= \frac{\lambda_{\varphi_{g_1}} \lambda_{\varphi_{g_2}} \lambda_{\varphi_{h_1}} \lambda_{\varphi_{h_2}}}{\lambda_{\varphi}^4} \cdot \frac{\tilde{\varepsilon}_{g_1, g_2}}{\tilde{\varepsilon}_{g_1', g_2'}} \cdot \frac{\tilde{\varepsilon}_{h_1, h_2}}{\tilde{\varepsilon}_{h_1', h_2'}} \\ & \quad \cdot (-1)^{\frac{1}{2}(N-N_{g_2'})(N-N_{g_2}-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_2') \right] \cdot \frac{\tilde{\varepsilon}_{g_1', g_2'} \tilde{\varepsilon}_{h_2', h_1'}}{\tilde{\varepsilon}_{g_1', h_1'}} \cdot [H_{g', h'}] \bar{v}_{g' h'} \\ &= (-1)^{\frac{1}{2}(N-N_{g_2})(N-N_{g_2}-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_2) \right] \cdot \frac{\tilde{\varepsilon}_{g_1, g_2} \tilde{\varepsilon}_{h_2, h_1}}{\tilde{\varepsilon}_{g_1, h_1}} \\ & \quad \cdot \left(\frac{\lambda_{\varphi_{g_2}}^2}{\lambda_{\varphi}^2} [H_{g', h'}] \right) \left(\frac{\lambda_{\varphi_{g_1}} \lambda_{\varphi_{h_1}}}{\lambda_{\varphi}^2} \cdot \frac{\tilde{\varepsilon}_{g_1, h_1}}{\tilde{\varepsilon}_{g_1', h_1'}} \bar{v}_{g' h'} \right) \\ &= c_{g,h} [\varphi^* H_{g,h}] \varphi^*(\bar{v}_{gh}) = \varphi^*(\bar{v}_g \circ \bar{v}_h), \end{aligned}$$

where we also used that

$$\tilde{\varepsilon}_{h_1, h_2} = (-1)^{(N-N_{h_1})(N-N_{h_2})} \tilde{\varepsilon}_{h_2, h_1}, \quad \tilde{\varepsilon}_{h_1', h_2'} = (-1)^{(N-N_{h_1'})(N-N_{h_2'})} \tilde{\varepsilon}_{h_2', h_1'}.$$

Hence, we proved the algebra structure \circ of \mathcal{A}' is $\text{Aut}(f, G)$ -invariant.

The G -twisted $\mathbb{Z}/2\mathbb{Z}$ -graded commutativity, axiom (ivb), is a direct consequence of Proposition 48 since $H_{g,h} = H_{h,g}$ and $g^*(\bar{v}_h) = \mathbf{e}[-\text{age}(g_2)] \cdot \bar{v}_h$ which follows from the calculation

$$\begin{aligned} g^*(\bar{v}_h) \vdash \zeta &= g^*(\bar{v}_h) \vdash (\mathbf{e}[-\text{age}(g)] g^*(\zeta)) = \mathbf{e}[-\text{age}(g)] \cdot g^*(\bar{\alpha}_h \omega_h) \\ &= \mathbf{e}[-\text{age}(g_2)] \cdot (\bar{\alpha}_h \omega_h) = (\mathbf{e}[-\text{age}(g_2)] \cdot \bar{v}_h) \vdash \zeta. \end{aligned}$$

We have finished the proof of the proposition. \square

We show the invariance of the bilinear form $J_{f,G}$ with respect to the product structure of \mathcal{A}' . We use the notation in Definition 27.

Proposition 62. *For a spanning pair (g, h) of elements of G_f , we have*

$$\begin{aligned} & J_{f,gh} \left(\bar{v}_g \vdash \omega_h, \left[\frac{1}{\mu_{f^{g \cap h}}} \text{hess}(f^{g \cap h}) \right] \omega_{(gh)^{-1}} \right) \\ &= (-1)^{(N-N_g)(N-N_h)} J_{f,h} \left(\omega_h, ((h^{-1})^* \bar{v}_g) \vdash \left(\left[\frac{1}{\mu_{f^{g \cap h}}} \text{hess}(f^{g \cap h}) \right] \omega_{(gh)^{-1}} \right) \right). \quad (4.55) \end{aligned}$$

As a consequence, the algebra \mathcal{A}' satisfies axiom (v).

Proof. Let (g_1, g_2, h_1, h_2) be the factorization of the spanning pair (g, h) . The LHS of the equation (4.55) is calculated as

$$\begin{aligned}
& J_{f,gh} \left(\bar{v}_g \vdash \omega_h, \left[\frac{1}{\mu_{fg \cap h}} \text{hess}(fg \cap h) \right] \omega_{(gh)^{-1}} \right) \\
&= \frac{1}{\bar{\alpha}_h} \cdot J_{f,gh} \left((\bar{v}_g \circ \bar{v}_h) \vdash \zeta, \left[\frac{1}{\mu_{fg \cap h}} \text{hess}(fg \cap h) \right] \omega_{(gh)^{-1}} \right) \\
&= \frac{\bar{\alpha}_{gh} \bar{c}_{g,h}}{\bar{\alpha}_h} \cdot J_{f,gh} \left(\omega_{gh}, \left[\frac{1}{\mu_{fg \cap h}} \text{hess}(fg \cap h) H_{g,h} \right] \omega_{(gh)^{-1}} \right) \\
&= \frac{\bar{\alpha}_{gh}}{\bar{\alpha}_h} \cdot (-1)^{\frac{1}{2}(N-N_{g_2})(N-N_{g_2}-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_2) \right] \cdot \frac{\tilde{\varepsilon}_{g_1, g_2} \tilde{\varepsilon}_{h_2, h_1}}{\tilde{\varepsilon}_{g_1, h_1}} \\
&\quad \cdot (-1)^{N-N_{gh}} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(gh) \right] \cdot |G| \\
&= \frac{\bar{\alpha}_{gh}}{\bar{\alpha}_h} \cdot (-1)^{\frac{1}{2}(N-N_{g_2})(N-N_{g_2}-1)+(N-N_{gh})} \\
&\quad \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_1) - \frac{1}{2} \text{age}(h_1) - \frac{1}{2} \text{age}(g_2) \right] \cdot \frac{\tilde{\varepsilon}_{g_1, g_2} \tilde{\varepsilon}_{h_2, h_1}}{\tilde{\varepsilon}_{g_1, h_1}} \cdot |G|.
\end{aligned}$$

On the other hand, the RHS of the equation (4.55) is calculated as

$$\begin{aligned}
& (-1)^{(N-N_g)(N-N_h)} \cdot J_{f,h} \left(\omega_h, ((h^{-1})^* \bar{v}_g) \vdash \left(\left[\frac{1}{\mu_{fg \cap h}} \text{hess}(fg \cap h) \right] \omega_{(gh)^{-1}} \right) \right) \\
&= \frac{1}{\bar{\alpha}_{(gh)^{-1}}} (-1)^{(N-N_g)(N-N_h)} \cdot \mathbf{e} [-\text{age}(h_2^{-1})] \\
&\quad \cdot J_{f,h} \left(\omega_h, \left(\left[\frac{1}{\mu_{fg \cap h}} \text{hess}(fg \cap h) \right] \bar{v}_g \circ \bar{v}_{(gh)^{-1}} \right) \vdash \zeta \right) \\
&= \frac{\bar{\alpha}_{h^{-1} C_{g, (gh)^{-1}}}}{\bar{\alpha}_{(gh)^{-1}}} (-1)^{(N-N_g)(N-N_h)} \cdot \mathbf{e} [-\text{age}(g_2)] \cdot J_{f,h} \left(\omega_h, \left[\frac{1}{\mu_{fg \cap h}} \text{hess}(fg \cap h) \right] \vdash \omega_{h^{-1}} \right) \\
&= \frac{\bar{\alpha}_{h^{-1}}}{\bar{\alpha}_{(gh)^{-1}}} (-1)^{(N-N_g)(N-N_h) + \frac{1}{2}(N-N_{g_1})(N-N_{g_1}-1)} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_1) - \text{age}(g_2) \right] \cdot \frac{\tilde{\varepsilon}_{g_2, g_1} \tilde{\varepsilon}_{g_1^{-1}, h_1}}{\tilde{\varepsilon}_{g_2, h_1}} \\
&\quad \cdot (-1)^{N-N_h} \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(h) \right] \cdot |G| \\
&= \frac{\bar{\alpha}_{h^{-1}}}{\bar{\alpha}_{(gh)^{-1}}} (-1)^{(N-N_g+1)(N-N_h) + \frac{1}{2}(N-N_{g_1})(N-N_{g_1}-1) - (N-N_{g_2}) + (N-N_{g_1})(N-N_{g_2})} \\
&\quad \cdot \mathbf{e} \left[-\frac{1}{2} \text{age}(g_1) - \frac{1}{2} \text{age}(h_1) - \frac{1}{2} \text{age}(g_2) \right] \cdot \frac{\tilde{\varepsilon}_{g_1, g_2} \tilde{\varepsilon}_{h_2, h_1}}{\tilde{\varepsilon}_{g_1, h_1}} \cdot |G|,
\end{aligned}$$

where we used that $\tilde{\varepsilon}_{g_2^{-1}, h_1}^{-1} = \tilde{\varepsilon}_{g_2^{-1}, h_1} = \tilde{\varepsilon}_{h_2, h_1}$ and $\tilde{\varepsilon}_{g_1^{-1}, h_1}^{-1} = \tilde{\varepsilon}_{g_1, h_1} = \tilde{\varepsilon}_{g_1^{-1}, h_1}^{-1}$. We have $\bar{\alpha}_{gh} \bar{\alpha}_{(gh)^{-1}} = (-1)^{\frac{1}{2}(N-N_{gh})(N-N_{gh}+1)}$ and $\bar{\alpha}_h \bar{\alpha}_{h^{-1}} = (-1)^{\frac{1}{2}(N-N_h)(N-N_h+1)}$ by (4.49). Hence, it follows from a direct calculation by the use of

$$\begin{aligned}
N - N_g &= (N - N_{g_1}) + (N - N_{g_2}), & N - N_h &= (N - N_{h_1}) + (N - N_{h_2}), \\
N - N_{gh} &= N - N_{g_1 g_2} = (N - N_{g_1}) + (N - N_{h_1}), & N_{g_2} &= N_{h_2},
\end{aligned}$$

(cf. Proposition 32) that

$$\begin{aligned} & \frac{1}{2}(N - N_{gh})(N - N_{gh} + 1) + \frac{1}{2}(N - N_{g_2})(N - N_{g_2} - 1) + (N - N_{gh}) \\ & - \frac{1}{2}(N - N_h)(N - N_h + 1) + (N - N_g + 1)(N - N_h) \\ & + \frac{1}{2}(N - N_{g_1})(N - N_{g_1} - 1) - (N - N_{g_2}) + (N - N_{g_1})(N - N_{g_2}) \\ & \equiv 0 \pmod{2}, \end{aligned}$$

which gives the equation (4.55)

For $X \in \mathcal{A}'_g$, $\omega \in \Omega'_{f,h}$, $\omega' \in \Omega'_{f,G}$, $J_{f,G}(X \vdash \omega, \omega')$ is non-zero only if $\omega' \in \Omega'_{f,(gh)^{-1}}$ and the pair (g, h) is a spanning pair. Note that $I_g \cup I_h \cup I_{gh} = I_{\text{id}}$ if and only if $I_h \cup I_{(gh)^{-1}} \cup I_{g^{-1}} = I_{\text{id}}$, which means the pair (g, h) is a spanning pair if and only if the pair $(h, (gh)^{-1})$ is so. Therefore, $J_{f,G}(X \vdash \omega, \omega')$ is non-zero if and only if $J_{f,G}(\omega, (h^{-1})^*X \vdash \omega')$ is so. It follows that the axiom (v) can be reduced to the equation (4.55). \square

The last axiom (vi) is trivially satisfied for \mathcal{A}' . Therefore, we have shown the existence of a G_f -twisted Jacobian algebra of f .

Moreover, it is easy to see the following

Proposition 63. *For each subgroup $G \subseteq G_f$, there exists a G -twisted Jacobian algebra of f .*

Proof. Consider the subspace \mathcal{A}'_G of \mathcal{A}' defined by

$$\mathcal{A}'_G := \bigoplus_{g \in G} \mathcal{A}'_g,$$

the restriction of the product structure map $\circ : \mathcal{A}' \otimes_{\mathbb{C}} \mathcal{A}' \rightarrow \mathcal{A}'$ to $\mathcal{A}'_G \otimes_{\mathbb{C}} \mathcal{A}'_G$ and the restriction of the \mathcal{A}' -module structure map $\vdash : \mathcal{A}' \otimes_{\mathbb{C}} \Omega'_{f,G_f} \rightarrow \mathcal{A}'$ to $\mathcal{A}'_G \otimes_{\mathbb{C}} \Omega'_{f,G}$. By the construction of these structures on \mathcal{A}' , it is almost obvious that they satisfy all the axioms in Definition 16. \square

We have finished the proof of Theorem 22.

5. ORBIFOLD JACOBIAN ALGEBRAS FOR ADE ORBIFOLDS

The classification of invertible polynomials in three variables giving ADE singularities and the subgroups of their maximal diagonal symmetries preserving the holomorphic volume form is given in Table 1 below (see also [ET13a] Section 8 Table 3).

As it is explained in Section 8 in [ET13a], one can describe explicitly the geometry of vanishing cycles for the holomorphic map $\widehat{f} : \widehat{\mathbb{C}^3/G} \rightarrow \mathbb{C}$. Here, $\widehat{\mathbb{C}^3/G}$ is a crepant resolution of \mathbb{C}^3/G and \widehat{f} is the convolution of the resolution map $\widehat{\mathbb{C}^3/G} \rightarrow \mathbb{C}^3/G$ and the induced one $f : \mathbb{C}^3/G \rightarrow \mathbb{C}$. Note that $\widehat{\mathbb{C}^3/G}$ is covered by some charts all isomorphic to \mathbb{C}^3 . When G respects one coordinate we only need to look at the resolutions of \mathbb{C}^2 given in [BaeKn]. For $G \cong \mathbb{Z}/2\mathbb{Z}$ acting $(z_i, z_j) \mapsto (-z_i, -z_j)$, we have $\mathbb{C}^3/G \cong \mathbb{C} \times \{z^2 = xy\} \subset \mathbb{C}^4$ by $x = z_i^2, y = z_j^2, z = z_i z_j$ and we have the two charts $\mathbb{C}^3 \rightarrow \mathbb{C}^4$;

$$(t, u, v) \mapsto (t, u, uv^2, uv) \text{ and } (t, u, v) \mapsto (t, u^2v, v, uv).$$

For $G \cong \mathbb{Z}/3\mathbb{Z} = \langle \frac{1}{3}(1, 2, 0) \rangle$, we have $\mathbb{C}^3/G \cong \mathbb{C} \times \{z^3 = xy\} \subset \mathbb{C}^4$ by

$$x = z_1^3, y = z_2^3, z = z_1 z_2$$

and we have the three charts $\mathbb{C}^3 \rightarrow \mathbb{C}^4$;

$$(t, u, v) \mapsto (t, u, u^2v^3, uv), (t, u, v) \mapsto (t, u^2v, uv^2, uv) \text{ and } (t, u, v) \mapsto (t, u^3v^2, v, uv).$$

We shall calculate the restriction of \widehat{f} on each chart based on the classification in Table 1.

Type	$f(x_1, x_2, x_3)$	$G_f \cap \mathrm{SL}(3; \mathbb{C})$	Singularity Type
I	$z_1^{2k+1} + z_2^2 + z_3^2, \quad k \geq 1$	$\langle \frac{1}{2}(0, 1, 1) \rangle$	A_{2k}
	$z_1^{2k} + z_2^2 + z_3^2, \quad k \geq 1$	$\langle \frac{1}{2}(0, 1, 1), \frac{1}{2}(1, 0, 1) \rangle$	A_{2k-1}
	$z_1^3 + z_2^3 + z_3^2$	$\langle \frac{1}{3}(1, 2, 0) \rangle$	D_4
	$z_1^4 + z_2^3 + z_3^2$	$\langle \frac{1}{2}(1, 0, 1) \rangle$	E_6
	$z_1^5 + z_2^3 + z_3^2$	$\{1\}$	E_8
II	$z_1^2 + z_2^2 + z_2 z_3^{2k}, \quad k \geq 1$	$\langle \frac{1}{2}(1, 0, 1) \rangle$	A_{4k-1}
	$z_1^2 + z_2^2 + z_2 z_3^{2k+1}, \quad k \geq 1$	$\langle \frac{1}{2}(0, 1, 1) \rangle$	A_{4k+1}
	$z_1^2 + z_2^{k-1} + z_2 z_3^2, \quad k \geq 4$	$\langle \frac{1}{2}(1, 0, 1) \rangle$	D_k
	$z_1^3 + z_2^2 + z_2 z_3^2$	$\{1\}$	E_6
	$z_1^2 + z_2^3 + z_2 z_3^3$	$\{1\}$	E_7
III	$z_1^2 + z_3 z_2^2 + z_2 z_3^{k+1}, \quad k \geq 1$	$\{1\}$	D_{2k+2}
IV	$z_1^k + z_1 z_2 + z_2 z_3^l, \quad k, l \geq 2$	$\{1\}$	A_{kl-1}
	$z_1^2 + z_1 z_2^k + z_2 z_3^2, \quad k \geq 2$	$\{1\}$	D_{2k+1}
V	$z_1 z_2 + z_2^k z_3 + z_3 z_1, \quad k, l \geq 1$	$\{1\}$	A_{kl}

TABLE 1. Classification of invertible polynomials giving ADE singularities and the groups of their diagonal symmetries preserving the holomorphic volume form.

1. For the pair

$$f := z_1^{k+1} + z_2^2 + z_3^2 \quad (k \geq 1), \quad G := \left\langle \frac{1}{2}(0, 1, 1) \right\rangle, \quad (5.1)$$

we have in the two charts

$$\widehat{f}(t, u, v) = t^{k+1} + u + uv^2 \quad \text{and} \quad \widehat{f}(t, u, v) = t^{k+1} + u^2 v + v. \quad (5.2)$$

Critical points of \widehat{f} are on the intersection of two charts.

2. For the pair

$$f := z_1^{2k} + z_2^2 + z_3^2 \quad (k \geq 1), \quad G := \left\langle \frac{1}{2}(1, 0, 1) \right\rangle, \quad (5.3)$$

we have in the two charts

$$\widehat{f}(t, u, v) = t^2 + u^k + uv^2 \quad \text{and} \quad \widehat{f}(t, u, v) = t^2 + u^2 k v^k + v. \quad (5.4)$$

Critical points of \widehat{f} are on the first chart.

3. For $k \geq 1$, set

$$f := z_1^{2k} + z_2^2 + z_3^2 \quad (k \geq 1), \quad G := \left\langle \frac{1}{2}(0, 1, 1), \frac{1}{2}(1, 0, 1) \right\rangle. \quad (5.5)$$

Here, since the resolution is not unique, we take A -Hilb \mathbb{C}^3 of [CR] where $A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We have $\mathbb{C}^3/G \cong \mathbb{C} \times \{z^3 = wxy\} \subset \mathbb{C}^4$ by $w = z_1^2, x = z_2^2, y = z_3^2, z = z_1 z_2 z_3$ and we have four charts $\mathbb{C}^3 \rightarrow \mathbb{C}^4$;

$$(t, u, v) \mapsto (t, u, tuv^2, tuv), \quad (t, u, v) \mapsto (t, tu^2 v, v, tuv), \\ (t, u, v) \mapsto (t^2 uv, u, v, tuv) \quad \text{and} \quad (t, u, v) \mapsto (tu, uv, tv, tuv).$$

Then we have in the four charts

$$\widehat{f}(t, u, v) = t^k + u + tuv^2, \quad \widehat{f}(t, u, v) = t^k + tu^2 v + v, \quad (5.6a)$$

$$\widehat{f}(t, u, v) = t^{2k} u^k v^k + u + v \quad \text{and} \quad \widehat{f}(t, u, v) = t^k u^k + uv + tv. \quad (5.6b)$$

Critical points of \widehat{f} are on the fourth chart.

4. For the pair

$$f := z_1^3 + z_2^3 + z_3^2, \quad G := \left\langle \frac{1}{3}(1, 2, 0) \right\rangle, \quad (5.7)$$

we have in the three charts

$$\widehat{f}(t, u, v) = t^2 + u + u^2v^3, \quad \widehat{f}(t, u, v) = t^2 + u^2v + uv^2 \quad \text{and} \quad \widehat{f}(t, u, v) = t^2 + u^3v^2 + v. \quad (5.8)$$

Critical points of \widehat{f} are on the second chart.

5. For the pair

$$f := z_1^4 + z_2^3 + z_3^2, \quad G := \left\langle \frac{1}{2}(1, 0, 1) \right\rangle, \quad (5.9)$$

we have in the two charts

$$\widehat{f}(t, u, v) = t^3 + u^2 + uv^2 \quad \text{and} \quad \widehat{f}(t, u, v) = t^3 + u^4v^2 + v. \quad (5.10)$$

Critical points of \widehat{f} are on the first chart.

6. For the pair

$$f := z_1^2 + z_2^2 + z_2z_3^{2k} \quad (k \geq 1), \quad G := \left\langle \frac{1}{2}(1, 0, 1) \right\rangle, \quad (5.11)$$

we have in the two charts

$$\widehat{f}(t, u, v) = t^2 + tu^kv^{2k} + u \quad \text{and} \quad \widehat{f}(t, u, v) = t^2 + tv^k + vu^2. \quad (5.12)$$

Critical points of \widehat{f} are on the second chart.

7. For the pair

$$f := z_1^2 + z_2^2 + z_2z_3^{2k+1} \quad (k \geq 1), \quad G := \left\langle \frac{1}{2}(0, 1, 1) \right\rangle, \quad (5.13)$$

we have in the two charts

$$\widehat{f}(t, u, v) = t^2 + u + u^{k+1}v^{2k+1} \quad \text{and} \quad \widehat{f}(t, u, v) = t^2 + vu^2 + uv^{k+1}. \quad (5.14)$$

Critical points of \widehat{f} are on the second chart.

8. For the pair

$$f := z_1^2 + z_2^{k-1} + z_2z_3^2 \quad (k \geq 4), \quad G := \left\langle \frac{1}{2}(1, 0, 1) \right\rangle, \quad (5.15)$$

we have in the two charts

$$\widehat{f}(t, u, v) = t^{k-1} + tuv^2 + u \quad \text{and} \quad \widehat{f}(t, u, v) = t^{k-1} + tv + vu^2. \quad (5.16)$$

Critical points of \widehat{f} are on the second chart.

To summarize, we observed that critical points of the map \widehat{f} are contained in one chart isomorphic to \mathbb{C}^3 . The restriction of \widehat{f} to the chart is given by \overline{f} defined in Table 2.

	$f(x_1, x_2, x_3)$	G	$\overline{f}(y_1, y_2, y_3)$
1.	$z_1^{k+1} + z_2^2 + z_3^2, \quad k \geq 1$	$\left\langle \frac{1}{2}(0, 1, 1) \right\rangle$	$y_1^{k+1} + y_2 + y_2y_3^2$
2.	$z_1^{2k} + z_2^2 + z_3^2, \quad k \geq 1$	$\left\langle \frac{1}{2}(1, 0, 1) \right\rangle$	$y_1^2 + y_2^k + y_2y_3^2$
3.	$z_1^{2k} + z_2^2 + z_3^2, \quad k \geq 1$	$\left\langle \frac{1}{2}(0, 1, 1), \frac{1}{2}(1, 0, 1) \right\rangle$	$y_1^k y_2^k + y_1y_3 + y_2y_3$
4.	$z_1^3 + z_2^3 + z_3^2$	$\left\langle \frac{1}{3}(1, 2, 0) \right\rangle$	$y_1^2 + y_3y_2^2 + y_2y_3^2$
5.	$z_1^4 + z_2^3 + z_3^2$	$\left\langle \frac{1}{2}(1, 0, 1) \right\rangle$	$y_1^3 + y_2^2 + y_2y_3^2$
6.	$z_1^2 + z_2^2 + z_2z_3^{2k}, \quad k \geq 1$	$\left\langle \frac{1}{2}(1, 0, 1) \right\rangle$	$y_1^2 + y_1y_2^k + y_2y_3^2$
7.	$z_1^2 + z_2^2 + z_2z_3^{2k+1}, \quad k \geq 1$	$\left\langle \frac{1}{2}(0, 1, 1) \right\rangle$	$y_1^2 + y_3y_2^2 + y_2y_3^{k+1}$
8.	$z_1^2 + z_2^{k-1} + z_2z_3^2, \quad k \geq 4$	$\left\langle \frac{1}{2}(1, 0, 1) \right\rangle$	$y_1^{k-1} + y_1y_2 + y_2y_3^2$

TABLE 2. $(f, G) \cong (\overline{f}, \{1\})$

Therefore, concerning the geometry of vanishing cycles, the pair (f, G) is equivalent to the pair $(\overline{f}, \{1\})$. Then, it is quite natural to expect that the orbifold Jacobian algebra $\text{Jac}(f, G)$ of (f, G) is isomorphic to the one $\text{Jac}(\overline{f}, \{1\})$ of the pair $(\overline{f}, \{1\})$, the usual Jacobian algebra $\text{Jac}(\overline{f})$ of \overline{f} , which is the following

Theorem 64. *There is an isomorphism of Frobenius algebras*

$$\text{Jac}(f, G) \cong \text{Jac}(\bar{f}, \{1\}) \quad (5.17)$$

for all f and \bar{f} in Table 2.

Proof. We shall give a proof of this theorem based on the classification in Table 2. Let the notation be as in Section 4. For each $g \in G$ let K_g be the maximal subgroup of G fixing $\text{Fix}(g)$. Define $e_g \in \text{Jac}(f, G)$ by $e_g := \frac{1}{|K_g|}v_g$, which is more natural element than v_g .

1. For $k \geq 1$, set

$$f := z_1^{k+1} + z_2^2 + z_3^2, \quad G := \langle g \rangle, \quad g := \frac{1}{2}(0, 1, 1), \quad (5.18)$$

$$\bar{f} := y_1^{k+1} + y_2 + y_2 y_3^2. \quad (5.19)$$

The Jacobian algebra $\text{Jac}(f, \{1\})$ and the bilinear form $J_{f, \{1\}}$ on $\Omega_{f, \{1\}}$ can be calculated as

$$\text{Jac}(f, \{1\}) = \mathbb{C}[z_1, z_2, z_3] / ((k+1)z_1^k, 2z_2, 2z_3) \cong \langle [1], [z_1], \dots, [z_1]^{k-1} \rangle_{\mathbb{C}}, \quad (5.20)$$

$$J_{f, \{1\}}([dz_1 \wedge dz_2 \wedge dz_3], [z_1^{k-1} dz_1 \wedge dz_2 \wedge dz_3]) = \frac{1}{4(k+1)}. \quad (5.21)$$

As a \mathbb{C} -module, the orbifold Jacobian algebra $\text{Jac}(f, G)$ is of the following form:

$$\text{Jac}(f, G) \cong \langle e_{\text{id}}, [z_1], \dots, [z_1]^{k-1} \rangle_{\mathbb{C}} \oplus \langle e_g, [z_1]e_g, \dots, [z_1]^{k-1}e_g \rangle_{\mathbb{C}}. \quad (5.22)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(f, G) = 2k$. The bilinear form $J_{f, G}$ on $\Omega_{f, G}$ can be calculated as

$$J_{f, \text{id}}(e_{\text{id}} \vdash \zeta, [z_1]^{k-1} \vdash \zeta) = J_{f, \text{id}}([dz_1 \wedge dz_2 \wedge dz_3], [z_1^{k-1} dz_1 \wedge dz_2 \wedge dz_3]) \quad (5.23a)$$

$$= 2 \cdot \frac{1}{4(k+1)} = \frac{1}{2(k+1)}, \quad (5.23b)$$

$$J_{f, g}(e_g \vdash \zeta, [z_1]^{k-1}e_g \vdash \zeta) = \frac{1}{4}J_{f, g}([dz_1], [z_1^{k-1} dz_1]) \quad (5.23c)$$

$$= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{k+1} = -\frac{1}{2(k+1)}, \quad (5.23d)$$

which imply the following relations in the orbifold Jacobian algebra $\text{Jac}(f, G)$:

$$[z_1]^k = 0, \quad e_g^2 = -e_{\text{id}}. \quad (5.24)$$

On the other hand, the Jacobian algebra $\text{Jac}(\bar{f}, \{1\})$ is given by

$$\text{Jac}(\bar{f}, \{1\}) = \mathbb{C}[y_1, y_2, y_3] / ((k+1)y_1^k, 1 + y_3^2, 2y_2 y_3) \quad (5.25)$$

$$\cong \mathbb{C}[y_1, y_3] / (y_1^k, y_3^2 + 1). \quad (5.26)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(\bar{f}, \{1\}) = 2k$. Therefore, we have an algebra isomorphism

$$\text{Jac}(\bar{f}, \{1\}) \xrightarrow{\cong} \text{Jac}(f, G), \quad [y_1] \mapsto [z_1], \quad [y_3] \mapsto e_g, \quad (5.27)$$

which is, moreover, an isomorphism of Frobenius algebras since we have

$$J_{\bar{f}, \{1\}}([dy_1 \wedge dy_2 \wedge dy_3], [y_1^{k-1} dy_1 \wedge dy_2 \wedge dy_3]) = \frac{1}{2(k+1)}. \quad (5.28)$$

2. For $k \geq 1$, set

$$f := z_1^{2k} + z_2^2 + z_3^2, \quad G := \langle g \rangle, \quad g := \frac{1}{2}(1, 0, 1), \quad (5.29)$$

$$\bar{f} := y_1^2 + y_2^k + y_2 y_3^2. \quad (5.30)$$

The Jacobian algebra $\text{Jac}(f, \{1\})$ and the bilinear form $J_{f, \{1\}}$ on $\Omega_{f, \{1\}}$ can be calculated as

$$\text{Jac}(f, \{1\}) = \mathbb{C}[z_1, z_2, z_3] / (2kz_1^{2k-1}, 2z_2, 2z_3) \cong \langle [1], [z_1], \dots, [z_1]^{2k-2} \rangle_{\mathbb{C}}, \quad (5.31)$$

$$J_{f, \{1\}}([dz_1 \wedge dz_2 \wedge dz_3], [z_1^{2k-2} dz_1 \wedge dz_2 \wedge dz_3]) = \frac{1}{8k}. \quad (5.32)$$

As a \mathbb{C} -module, the orbifold Jacobian algebra $\text{Jac}(f, G)$ is of the following form:

$$\text{Jac}(f, G) \cong \langle e_{\text{id}}, [z_1^2], \dots, [z_1^{2k-1}] \rangle_{\mathbb{C}} \oplus \langle e_g \rangle_{\mathbb{C}}. \quad (5.33)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(f, G) = k + 1$. The bilinear form $J_{f, G}$ on $\Omega_{f, G}$ can be calculated as

$$J_{f, \text{id}}(e_{\text{id}} \vdash \zeta, [z_1^{2k-1}] \vdash \zeta) = J_{f, \text{id}}([dz_1 \wedge dz_2 \wedge dz_3], [z_1^{2k-2} dz_1 \wedge dz_2 \wedge dz_3]) \quad (5.34a)$$

$$= 2 \cdot \frac{1}{8k} = \frac{1}{4k}, \quad (5.34b)$$

$$J_{f, g}(e_g \vdash \zeta, e_g \vdash \zeta) = \frac{1}{4} J_{f, g}([dz_2], [dz_2]) \quad (5.34c)$$

$$= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{2} = -\frac{1}{4}, \quad (5.34d)$$

which imply the following relations in the orbifold Jacobian algebra $\text{Jac}(f, G)$:

$$[z_1^2] \circ e_g = 0, \quad e_g^2 = -k[z_1^{2k-1}]. \quad (5.35)$$

On the other hand, the Jacobian algebra $\text{Jac}(\bar{f}, \{1\})$ is given by

$$\text{Jac}(\bar{f}, \{1\}) = \mathbb{C}[y_1, y_2, y_3] / (2y_1, ky_2^{k-1} + y_3^2, 2y_2y_3) \quad (5.36)$$

$$\cong \mathbb{C}[y_2, y_3] / (ky_2^{k-1} + y_3^2, y_2y_3). \quad (5.37)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(\bar{f}, \{1\}) = k + 1$. Therefore, we have an algebra isomorphism

$$\text{Jac}(\bar{f}, \{1\}) \xrightarrow{\cong} \text{Jac}(f, G), \quad [y_2] \mapsto [z_1^2], \quad [y_3] \mapsto e_g, \quad (5.38)$$

which is, moreover, an isomorphism of Frobenius algebras since we have

$$J_{\bar{f}, \{1\}}([dy_1 \wedge dy_2 \wedge dy_3], [y_2^{k-1} dy_1 \wedge dy_2 \wedge dy_3]) = \frac{1}{4k}. \quad (5.39)$$

3. For $k \geq 1$, set

$$f := z_1^{2k} + z_2^2 + z_3^2, \quad G := \langle g, h \rangle, \quad g := \frac{1}{2}(0, 1, 1), \quad h := \frac{1}{2}(1, 0, 1), \quad (5.40)$$

$$\bar{f} := y_1^k y_2^k + y_1 y_3 + y_2 y_3. \quad (5.41)$$

The Jacobian algebra $\text{Jac}(f, \{1\})$ and the bilinear form $J_{f, \{1\}}$ on $\Omega_{f, \{1\}}$ can be calculated as

$$\text{Jac}(f, \{1\}) = \mathbb{C}[z_1, z_2, z_3] / (2kz_1^{2k-1}, 2z_2, 2z_3) \cong \langle [1], [z_1], \dots, [z_1]^{2k-2} \rangle_{\mathbb{C}}, \quad (5.42)$$

$$J_{f, \{1\}}([dz_1 \wedge dz_2 \wedge dz_3], [z_1^{2k-2} dz_1 \wedge dz_2 \wedge dz_3]) = \frac{1}{8k}. \quad (5.43)$$

As a \mathbb{C} -module, the orbifold Jacobian algebra $\text{Jac}(f, G)$ is of the following form:

$$\text{Jac}(f, G) \cong \langle e_{\text{id}}, [z_1^2], \dots, [z_1^{2k-1}] \rangle_{\mathbb{C}} \oplus \langle e'_g, [z_1^2]e'_g, \dots, [z_1^{2k-2}]e'_g \rangle_{\mathbb{C}}, \quad (5.44)$$

where $e'_g := [z_1]e_g$ since $\text{Jac}(f, h) = \{0\}$ and $\text{Jac}(f, gh) = \{0\}$.

Note that $\dim_{\mathbb{C}} \text{Jac}(f, G) = 2k - 1$.

The bilinear form $J_{f, G}$ on $\Omega_{f, G}$ can be calculated as

$$J_{f, \text{id}}(e_{\text{id}} \vdash \zeta, [z_1^{2k-1}] \vdash \zeta) = J_{f, \text{id}}([dz_1 \wedge dz_2 \wedge dz_3], [z_1^{2k-2} dz_1 \wedge dz_2 \wedge dz_3]) \quad (5.45a)$$

$$= 4 \cdot \frac{1}{8k} = \frac{1}{2k}, \quad (5.45b)$$

$$J_{f, g}(e'_g \vdash \zeta, [z_1^{2k-2}]e'_g \vdash \zeta) = \frac{1}{4} J_{f, g}([z_1^2 dz_1], [z_1^{2k-3} dz_1]) \quad (5.45c)$$

$$= \frac{1}{4} \cdot (-1) \cdot 4 \cdot \frac{1}{2k} = -\frac{1}{2k}, \quad (5.45d)$$

which imply the following relations in the orbifold Jacobian algebra $\text{Jac}(f, G)$:

$$[z_1^{2k-1}] \circ e'_g = 0, \quad (e'_g)^2 = -[z_1^2]. \quad (5.46)$$

On the other hand, the Jacobian algebra $\text{Jac}(\bar{f}, \{1\})$ is given by

$$\text{Jac}(\bar{f}, \{1\}) = \mathbb{C}[y_1, y_2, y_3] / (ky_1^{k-1}y_2^k + y_3, ky_1^k y_2^{k-1} + y_3, y_1 + y_2). \quad (5.47)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(\bar{f}, \{1\}) = 2k - 1$. Therefore, we have an algebra isomorphism

$$\text{Jac}(\bar{f}, \{1\}) \xrightarrow{\cong} \text{Jac}(f, G), \quad [y_1 y_2] \mapsto [z_1^2], [y_1] \mapsto e'_g, \quad (5.48)$$

which is, moreover, an isomorphism of Frobenius algebras since we have

$$J_{\bar{f}, \{1\}}([dy_1 \wedge dy_2 \wedge dy_3], [y_1^{k-1} y_2^{k-1} dy_1 \wedge dy_2 \wedge dy_3]) = \frac{1}{2k}. \quad (5.49)$$

4. Set

$$f := z_1^3 + z_2^3 + z_3^2, \quad G := \langle g \rangle, \quad g := \frac{1}{3}(1, 2, 0), \quad (5.50)$$

$$\bar{f} := y_1^2 + y_3 y_2^2 + y_2 y_3^2. \quad (5.51)$$

The Jacobian algebra $\text{Jac}(f, \{1\})$ and the bilinear form $J_{f, \{1\}}$ on $\Omega_{f, \{1\}}$ can be calculated as

$$\text{Jac}(f, \{1\}) \cong \mathbb{C}[z_1, z_2, z_3] / (3z_1^2, 3z_2^2, 2z_3) \cong \langle [1], [z_1], [z_2], [z_1 z_2] \rangle_{\mathbb{C}}, \quad (5.52)$$

$$J_{f, \{1\}}([dz_1 \wedge dz_2 \wedge dz_3], [z_1 z_2 dz_1 \wedge dz_2 \wedge dz_3]) = \frac{1}{18}. \quad (5.53)$$

As a \mathbb{C} -module, the orbifold Jacobian algebra $\text{Jac}(f, G)$ is of the following form:

$$\text{Jac}(f, G) \cong \langle e_{\text{id}}, [z_1 z_2] \rangle_{\mathbb{C}} \oplus \langle e_g, e_{g^{-1}} \rangle_{\mathbb{C}}. \quad (5.54)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(f, G) = 4$. The bilinear form $J_{f, G}$ on $\Omega_{f, G}$ can be calculated as

$$J_{f, \text{id}}(e_{\text{id}} \vdash \zeta, [z_1 z_2] \vdash \zeta) = J_{f, \text{id}}([dz_1 \wedge dz_2 \wedge dz_3], [z_1 z_2 dz_1 \wedge dz_2 \wedge dz_3]) \quad (5.55a)$$

$$= 3 \cdot \frac{1}{18} = \frac{1}{6}, \quad (5.55b)$$

$$J_{f, g}(e_g \vdash \zeta, e_{g^{-1}} \vdash \zeta) = \frac{1}{9} J_{f, g}([dz_3], [dz_3]) \quad (5.55c)$$

$$= \frac{1}{9} \cdot (-1) \cdot 3 \cdot \frac{1}{2} = -\frac{1}{6}, \quad (5.55d)$$

which imply the following relations in the orbifold Jacobian algebra $\text{Jac}(f, G)$:

$$e_g^2 = 0, \quad e_{g^{-1}}^2 = 0, \quad e_g \circ e_{g^{-1}} = -[z_1 z_2]. \quad (5.56)$$

On the other hand, the Jacobian algebra $\text{Jac}(\bar{f}, \{1\})$ is given by

$$\text{Jac}(\bar{f}, \{1\}) = \mathbb{C}[y_1, y_2, y_3] / (2y_1, 2y_3 y_2 + y_3^2, y_2^2 + 2y_2 y_3) \quad (5.57)$$

$$\cong \mathbb{C}[y_2, y_3] / (2y_3 y_2 + y_3^2, y_2^2 + 2y_2 y_3). \quad (5.58)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(\bar{f}, \{1\}) = 4$. Therefore, we have an algebra isomorphism

$$\begin{aligned} \text{Jac}(\bar{f}, \{1\}) &\xrightarrow{\cong} \text{Jac}(f, G), \\ [y_2] &\mapsto e^{\frac{2\pi\sqrt{-1}}{3}} e_g + e^{\frac{4\pi\sqrt{-1}}{3}} e_{g^{-1}}, \quad [y_3] \mapsto e^{\frac{4\pi\sqrt{-1}}{3}} e_g + e^{\frac{2\pi\sqrt{-1}}{3}} e_{g^{-1}}, \end{aligned} \quad (5.59)$$

which is, moreover, an isomorphism of Frobenius algebras since we have

$$J_{\bar{f}, \{1\}}([dy_1 \wedge dy_2 \wedge dy_3], [y_2 y_3 dy_1 \wedge dy_2 \wedge dy_3]) = \frac{1}{6}. \quad (5.60)$$

5. Set

$$f := z_1^4 + z_2^3 + z_3^2, \quad G := \langle g \rangle, \quad g := \frac{1}{2}(1, 0, 1), \quad (5.61)$$

$$\bar{f} := y_1^3 + y_2^2 + y_2 y_3^2. \quad (5.62)$$

The Jacobian algebra $\text{Jac}(f, \{1\})$ and the bilinear form $J_{f, \{1\}}$ on $\Omega_{f, \{1\}}$ can be calculated as

$$\text{Jac}(f, \{1\}) = \mathbb{C}[z_1, z_2, z_3] / (4z_1^3, 3z_2^2, 2z_3) \cong \langle [1], [z_1], [z_2], [z_1^2], [z_1 z_2], [z_1^2 z_2] \rangle_{\mathbb{C}}, \quad (5.63)$$

$$J_{f,\{1\}}([dz_1 \wedge dz_2 \wedge dz_3], [z_1^2 z_2 dz_1 \wedge dz_2 \wedge dz_3]) = \frac{1}{24}. \quad (5.64)$$

As a \mathbb{C} -module, the orbifold Jacobian algebra $\text{Jac}(f, G)$ is of the following form:

$$\text{Jac}(f, G) \cong \langle e_{\text{id}}, [z_2], [z_1^2], [z_1^2 z_2] \rangle_{\mathbb{C}} \oplus \langle e_g, [z_2] e_g \rangle_{\mathbb{C}}. \quad (5.65)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(f, G) = 6$. The bilinear form $J_{f,G}$ on $\Omega_{f,G}$ can be calculated as

$$J_{f,\text{id}}(e_{\text{id}} \vdash \zeta, [z_1^2 z_2] \vdash \zeta) = J_{f,\text{id}}([dz_1 \wedge dz_2 \wedge dz_3], [z_1^2 z_2 dz_1 \wedge dz_2 \wedge dz_3]) \quad (5.66a)$$

$$= 2 \cdot \frac{1}{24} = \frac{1}{12}, \quad (5.66b)$$

$$J_{f,g}(e_g \vdash \zeta, [z_2] e_g \vdash \zeta) = \frac{1}{4} J_{f,g}([dz_2], [z_2 dz_2]) \quad (5.66c)$$

$$= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{3} = -\frac{1}{6}, \quad (5.66d)$$

which imply the following relations in the orbifold Jacobian algebra $\text{Jac}(f, G)$:

$$[z_2]^2 = 0, \quad [z_1^2] \circ e_g = 0, \quad e_g^2 = -2[z_1^2]. \quad (5.67)$$

On the other hand, the Jacobian algebra $\text{Jac}(\bar{f}, \{1\})$ is given by

$$\text{Jac}(\bar{f}, \{1\}) = \mathbb{C}[y_1, y_2, y_3] / (3y_1^2, 2y_2 + y_2^2, 2y_2 y_3). \quad (5.68)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(\bar{f}, \{1\}) = 6$. Therefore, we have an algebra isomorphism

$$\text{Jac}(\bar{f}, \{1\}) \xrightarrow{\cong} \text{Jac}(f, G), \quad [y_1] \mapsto [z_2], \quad [y_2] \mapsto [z_1^2], \quad [y_3] \mapsto e_g, \quad (5.69)$$

which is, moreover, an isomorphism of Frobenius algebras since we have

$$J_{\bar{f},\{1\}}([dy_1 \wedge dy_2 \wedge dy_3], [y_1 y_2 dy_1 \wedge dy_2 \wedge dy_3]) = \frac{1}{12}. \quad (5.70)$$

6. For $k \geq 1$, set

$$f := z_1^2 + z_2^2 + z_2 z_3^{2k}, \quad G := \langle g \rangle, \quad g := \frac{1}{2}(1, 0, 1), \quad (5.71)$$

$$\bar{f} := y_1^2 + y_1 y_2^k + y_2 y_3^2. \quad (5.72)$$

The Jacobian algebra $\text{Jac}(f, \{1\})$ and the bilinear form $J_{f,\{1\}}$ on $\Omega_{f,\{1\}}$ can be calculated as

$$\text{Jac}(f, \{1\}) = \mathbb{C}[z_1, z_2, z_3] / (2z_1, 2z_2 + z_3^{2k}, 2kz_2 z_3^{2k-1}) \quad (5.73)$$

$$\cong \langle [1], [z_3], \dots, [z_3]^{2k-1}, [z_2], [z_2 z_3], \dots, [z_2 z_3^{2k-2}] \rangle_{\mathbb{C}}, \quad (5.74)$$

$$J_{f,\{1\}}([dz_1 \wedge dz_2 \wedge dz_3], [z_2 z_3^{2k-2} dz_1 \wedge dz_2 \wedge dz_3]) = \frac{1}{8k}. \quad (5.75)$$

As a \mathbb{C} -module, the orbifold Jacobian algebra $\text{Jac}(f, G)$ is of the following form:

$$\text{Jac}(f, G) \cong \langle e_{\text{id}}, [z_3^2], \dots, [z_3]^{2k-1}, [z_2], [z_2] \cdot [z_3^2], \dots, [z_2] \cdot [z_3]^{2k-1} \rangle_{\mathbb{C}} \oplus \langle e_g \rangle_{\mathbb{C}}. \quad (5.76)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(f, G) = 2k + 1$. The bilinear form $J_{f,G}$ on $\Omega_{f,G}$ can be calculated as

$$J_{f,\text{id}}(e_{\text{id}} \vdash \zeta, [z_2] \cdot [z_3]^{2k-1} \vdash \zeta) = J_{f,\text{id}}([dz_1 \wedge dz_2 \wedge dz_3], [z_2 z_3^{2k-2} dz_1 \wedge dz_2 \wedge dz_3]) \quad (5.77a)$$

$$= 2 \cdot \frac{1}{8k} = \frac{1}{4k}, \quad (5.77b)$$

$$J_{f,g}(e_g \vdash \zeta, e_g \vdash \zeta) = \frac{1}{4} J_{f,g}([dz_2], [dz_2]) \quad (5.77c)$$

$$= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{2} = -\frac{1}{4}, \quad (5.77d)$$

which imply the following relations in the orbifold Jacobian algebra $\text{Jac}(f, G)$:

$$2[z_2] + [z_3]^{2k} = 0, \quad [z_3^2] \circ e_g = 0, \quad e_g^2 = -k[z_2] \cdot [z_3]^{2k-1}. \quad (5.78)$$

On the other hand, the Jacobian algebra $\text{Jac}(\bar{f}, \{1\})$ is given by

$$\text{Jac}(\bar{f}, \{1\}) = \mathbb{C}[y_1, y_2, y_3] / \left(2y_1 + y_2^k, ky_1y_2^{k-1} + y_3^2, 2y_2y_3 \right). \quad (5.79)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(\bar{f}, \{1\}) = 2k + 1$. Therefore, we have an algebra isomorphism

$$\text{Jac}(\bar{f}, \{1\}) \xrightarrow{\cong} \text{Jac}(f, G), \quad [y_1] \mapsto [z_2], [y_2] \mapsto [z_3^2], [y_3] \mapsto e_g, \quad (5.80)$$

which is, moreover, an isomorphism of Frobenius algebras since we have

$$J_{\bar{f}, \{1\}} \left([dy_1 \wedge dy_2 \wedge dy_3], [y_1y_2^{k-1}dy_1 \wedge dy_2 \wedge dy_3] \right) = \frac{1}{4k}. \quad (5.81)$$

7. For $k \geq 1$, set

$$f := z_1^2 + z_2^2 + z_2z_3^{2k+1}, \quad G := \langle g \rangle, \quad g := \frac{1}{2}(0, 1, 1), \quad (5.82)$$

$$\bar{f} := y_1^2 + y_3y_2^2 + y_2y_3^{k+1}. \quad (5.83)$$

The Jacobian algebra $\text{Jac}(f, \{1\})$ and the bilinear form $J_{f, \{1\}}$ on $\Omega_{f, \{1\}}$ can be calculated as

$$\text{Jac}(f, \{1\}) = \mathbb{C}[z_1, z_2, z_3] / \left(2z_1, 2z_2 + z_3^{2k+1}, (2k+1)z_2z_3^{2k} \right) \quad (5.84)$$

$$\cong \left\langle [1], [z_3], \dots, [z_3]^{2k}, [z_2], [z_2z_3], \dots, [z_2z_3^{2k-1}] \right\rangle_{\mathbb{C}}, \quad (5.85)$$

$$J_{f, \{1\}} \left([dz_1 \wedge dz_2 \wedge dz_3], [z_2z_3^{2k-1}dz_1 \wedge dz_2 \wedge dz_3] \right) = \frac{1}{4(2k+1)}. \quad (5.86)$$

As a \mathbb{C} -module, the orbifold Jacobian algebra $\text{Jac}(f, G)$ is of the following form:

$$\text{Jac}(f, G) \cong \left\langle e_{\text{id}}, [z_3^2], \dots, [z_3]^{2k}, [z_2z_3], [z_2z_3] \cdot [z_3^2], \dots, [z_2z_3] \cdot [z_3]^{2k-1} \right\rangle_{\mathbb{C}} \oplus \langle e_g \rangle_{\mathbb{C}}. \quad (5.87)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(f, G) = 2(k+1)$. The bilinear form $J_{f, G}$ on $\Omega_{f, G}$ can be calculated as

$$J_{f, \text{id}} \left(e_{\text{id}} \vdash \zeta, [z_2z_3] \cdot [z_3]^{2k-1} \vdash \zeta \right) = J_{f, \text{id}} \left([dz_1 \wedge dz_2 \wedge dz_3], [z_2z_3^{2k-1}dz_1 \wedge dz_2 \wedge dz_3] \right) \quad (5.88a)$$

$$= 2 \cdot \frac{1}{4(2k+1)} = \frac{1}{2(2k+1)}, \quad (5.88b)$$

$$J_{f, g} \left(e_g \vdash \zeta, e_g \vdash \zeta \right) = \frac{1}{4} J_{f, g} \left([dz_1], [dz_1] \right) \quad (5.88c)$$

$$= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{2} = -\frac{1}{4}, \quad (5.88d)$$

which imply the following relations in the orbifold Jacobian algebra $\text{Jac}(f, G)$:

$$2[z_2z_3] + [z_3]^{2k+1} = 0, \quad [z_3^2] \circ e_g = 0, \quad e_g^2 = -\frac{2k+1}{2}[z_2z_3][z_3]^{k-1}. \quad (5.89)$$

On the other hand, the Jacobian algebra $\text{Jac}(\bar{f}, \{1\})$ is given by

$$\text{Jac}(\bar{f}, \{1\}) = \mathbb{C}[y_1, y_2, y_3] / \left(2y_1, 2y_3y_2 + y_3^{k+1}, y_2^2 + (k+1)y_2y_3^k \right) \quad (5.90)$$

$$\cong \mathbb{C}[y_2, y_3] / \left(2y_3y_2 + y_3^{k+1}, y_2^2 + (k+1)y_2y_3^k \right). \quad (5.91)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(\bar{f}, \{1\}) = 2(k+1)$. Therefore, we have an algebra isomorphism

$$\text{Jac}(\bar{f}, \{1\}) \xrightarrow{\cong} \text{Jac}(f, G), \quad [y_2] \mapsto e_g - \frac{1}{2}[z_3^2]^k, [y_3] \mapsto [z_3^2], \quad (5.92)$$

which is, moreover, an isomorphism of Frobenius algebras since we have

$$J_{\bar{f}, \{1\}} \left([dy_1 \wedge dy_2 \wedge dy_3], [y_2y_3^kdy_1 \wedge dy_2 \wedge dy_3] \right) = \frac{1}{2(2k+1)}. \quad (5.93)$$

8. For $k \geq 4$, set

$$f := z_1^2 + z_2^{k-1} + z_2z_3^2, \quad G := \langle g \rangle, \quad g := \frac{1}{2}(1, 0, 1), \quad (5.94)$$

$$\bar{f} := y_1^{k-1} + y_1y_2 + y_2y_3^2. \quad (5.95)$$

The Jacobian algebra $\text{Jac}(f, \{1\})$ and the bilinear form $J_{f, \{1\}}$ on $\Omega_{f, \{1\}}$ can be calculated as

$$\text{Jac}(f, \{1\}) = \mathbb{C}[z_1, z_2, z_3] / \left(2z_1, (k-1)z_2^{k-2} + z_3^2, 2z_2z_3 \right) \quad (5.96)$$

$$\cong \left\langle [1], [z_2], \dots, [z_2]^{k-2}, [z_3] \right\rangle_{\mathbb{C}}, \quad (5.97)$$

$$J_{f, \{1\}} \left([dz_1 \wedge dz_2 \wedge dz_3], [z_2^{k-2} dz_1 \wedge dz_2 \wedge dz_3] \right) = \frac{1}{4(k-1)}. \quad (5.98)$$

As a \mathbb{C} -module, the orbifold Jacobian algebra $\text{Jac}(f, G)$ is of the following form:

$$\text{Jac}(f, G) \cong \left\langle e_{\text{id}}, [z_2], \dots, [z_2]^{k-2} \right\rangle_{\mathbb{C}} \oplus \left\langle e_g, [z_2]e_g, \dots, [z_2]^{k-3}e_g \right\rangle_{\mathbb{C}}. \quad (5.99)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(f, G) = 2k - 3$. The bilinear form $J_{f, G}$ on $\Omega_{f, G}$ can be calculated as

$$J_{f, \text{id}} \left(e_{\text{id}} \vdash \zeta, [z_2]^{k-2} \vdash \zeta \right) = J_{f, \text{id}} \left([dz_1 \wedge dz_2 \wedge dz_3], [z_2^{k-2} dz_1 \wedge dz_2 \wedge dz_3] \right) \quad (5.100a)$$

$$= 2 \cdot \frac{1}{4(k-1)} = \frac{1}{2(k-1)}, \quad (5.100b)$$

$$J_{f, g} \left(e_g \vdash \zeta, [z_2]^{k-3} e_g \vdash \zeta \right) = \frac{1}{4} J_{f, g} \left([dz_2], [z_2^{k-3} dz_2] \right) \quad (5.100c)$$

$$= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{k-1} = -\frac{1}{2(k-1)}, \quad (5.100d)$$

which imply the following relations in the orbifold Jacobian algebra $\text{Jac}(f, G)$:

$$[z_2]^{k-2} \circ e_g = 0, \quad e_g^2 = -[z_2]. \quad (5.101)$$

On the other hand, the Jacobian algebra $\text{Jac}(\bar{f}, \{1\})$ is given by

$$\text{Jac}(\bar{f}, \{1\}) = \mathbb{C}[y_1, y_2, y_3] / \left((k-1)y_1^{k-2} + y_2, y_1 + y_3^2, 2y_2y_3 \right). \quad (5.102)$$

Note that $\dim_{\mathbb{C}} \text{Jac}(\bar{f}, \{1\}) = 2k - 3$. Therefore, we have an algebra isomorphism

$$\text{Jac}(\bar{f}, \{1\}) \xrightarrow{\cong} \text{Jac}(f, G), \quad [y_1] \mapsto [z_2], \quad [y_2] \mapsto -(k-1)[z_2]^{k-2}, \quad [y_3] \mapsto e_g, \quad (5.103)$$

which is, moreover, an isomorphism of Frobenius algebras since we have

$$J_{\bar{f}, \{1\}} \left([dy_1 \wedge dy_2 \wedge dy_3], [y_1^{k-2} dy_1 \wedge dy_2 \wedge dy_3] \right) = \frac{1}{2(k-1)}. \quad (5.104)$$

We finished the proof of Theorem 64. \square

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A. BASALAEV, FACULTY OF MATHEMATICS, NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS,
USACHEVA STR., 6, 119048 MOSCOW, RUSSIAN FEDERATION, AND
SKOLKOVO INSTITUTE OF SCIENCE AND TECHNOLOGY, NOBELYA STR., 3, 121205 MOSCOW, RUSSIAN FEDERATION
Email address: a.basalaev@skoltech.ru

A. TAKAHASHI, DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, TOYONAKA
OSAKA, 560-0043, JAPAN
Email address: takahashi@math.sci.osaka-u.ac.jp

E. WERNER, LEIBNIZ UNIVERSITÄT HANNOVER, WELFENGARTEN 1, 30167 HANNOVER, GERMANY
Email address: werner.elisabeth@math.uni-hannover.de