
SMOOTH RIGIDITY AND REMEZ INEQUALITIES VIA TOPOLOGY OF LEVEL SETS

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ABSTRACT. A smooth rigidity inequality provides an explicit lower bound for the $(d + 1)$ -st derivatives of a smooth function f , which holds, if f exhibits certain patterns, forbidden for polynomials of degree d . The main goal of the present paper is twofold: first, we provide an overview of some recent results and questions related to smooth rigidity, which recently were obtained in Singularity Theory, in Approximation Theory, and in Whitney smooth extensions. Second, we prove some new results, specifically, a new Remez-type inequality, and on this base we obtain a new rigidity inequality. In both parts of the paper we stress the topology of the level sets, as the input information.

Here is a very informal statement of the main new result of this paper: the classical Remez-type inequality compares the maxima of a polynomial on the unit ball B and on its sub-domain Z . The answer is given in terms of the volume of Z . We replace Z by its boundary (whose volume is zero), but require Z to have sufficiently many connected components. Then the answer is given in terms of the minimal volume of the connected components of Z .

1. INTRODUCTION

Let $f(x)$ be a smooth function on the unit n -dimensional ball B^n . A “rigidity inequality” for f is an explicit lower bound for the $(d + 1)$ -st derivative of f , which holds, if f exhibits certain patterns, forbidden for polynomials of degree d .

We expect rigidity inequalities to be valid for those polynomial behavior patterns, which are stable with respect to smooth approximations. At present many such important patterns are known (see [8, 14, 19, 16]). However, translation of the known “near-polynomiality” results into “rigidity inequalities” usually is not straightforward, and many new questions arise.

Smooth rigidity inequalities naturally form a certain domain of interrelated results and questions in Smooth Analysis. They are closely related to some other important mathematical fields, including Singularity Theory, Approximation Theory, Real Algebraic Geometry, and Whitney extension problems.

The main goal of the present paper is twofold: first, we provide an overview of some recent results and questions related to smooth rigidity. We start with Remez-type inequalities ([15, 16, 17]), continue with recent exciting developments in the classical Whitney smooth extension theory ([3, 6, 7]), and conclude with a recent important development in global Singularity Theory, achieved in [8].

Second, we prove some new results. This includes a new Remez-type inequality. On this base, via the results of [17], we obtain a new Smooth rigidity inequality. We also want to illustrate in more detail some results of [8] and their consequences for smooth rigidity. For this purpose we give a direct proof of one very special case of the general results of [8]. On this base we provide also the corresponding rigidity inequality. In all these new (or almost new) results the stress is on the topology of the level sets, as the input information.

Here is our main new result: let B^n be the unit n -dimensional ball. For a given integer d let $Z \subset B^n$ be a smooth compact hypersurface with $N = (d - 1)^n + 1$ connected components Z_j . Let μ_j be the n -volume of the interior of Z_j , and put $\mu = \min \mu_j$, $j = 1, \dots, N$. Then for each polynomial P of degree d on \mathbb{R}^n we have

$$\frac{\max_{B^n} |P|}{\max_Z |P|} \leq \left(\frac{4n}{\mu}\right)^d.$$

As a consequence, we provide an explicit lower bound for the $(d + 1)$ -st derivatives of any smooth function f , which vanishes on Z , while being of order 1 on B^n (smooth rigidity):

$$\|f^{(d+1)}\| \geq \frac{1}{(d + 1)!} \left(\frac{4n}{\mu}\right)^d.$$

We also provide an interpretation, in terms of smooth rigidity, of one of the simplest versions of the results in [8].

The paper is organized as follows: Sections 2 - 5 form a review part of the paper. Specifically, in Sections 2 and 3 we provide an overview of some results of [17]. In Section 4 we shortly outline some connections of smooth rigidity with the recent important progress in the Whitney smooth extension problem ([3, 6, 7]). In Section 5 we provide a short overview of the results of [8].

Sections 6 and 7 provide new results: in Section 6 we obtain a new Remez-type inequality, which uses both the topological and measure information. On this base a corresponding rigidity inequality is obtained. In Section 7 we give a proof of one result in the spirit of [8], and provide on this base the corresponding rigidity inequality.

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2. SMOOTH RIGIDITY - SOME BACKGROUND

In this section we summarize, following [17], some basic facts concerning smooth rigidity. One of possible specific settings of the Smooth rigidity problem was proposed in [17]. It assumes as an input data a certain closed subset Z of the zeroes set $Y(f)$ of f . The following definition was given in [17]:

Let $f : B^n \rightarrow \mathbb{R}$ be a $d + 1$ times continuously differentiable function on the unit closed ball $B^n \subset \mathbb{R}^n$. For $l = 0, 1, \dots, d + 1$ put

$$M_l(f) = \max_{z \in B^n} \|f^{(l)}(z)\|,$$

where the norm of the l -th derivative $f^{(l)}(z)$ of f is defined as the sum of the absolute values of all the partial derivatives of f of order l .

For $Z \subset B^n$ let $W_d(Z)$ denote the set of C^{d+1} smooth functions $f(z)$ on B^n , vanishing on Z , with $M_0(f) = 1$.

Definition 2.1. For $Z \subset B^n$ we define the d -th rigidity constant $\mathcal{R}G_d(Z)$ as

$$\mathcal{R}G_d(Z) = \inf_{f \in W_d(Z)} M_{d+1}(f).$$

By this definition we get immediately $M_{d+1}(f) \geq \mathcal{R}G_d(Z)$ for any $f(z)$ on B^n , vanishing on Z , with $M_0(f) = 1$. Our goal is to estimate $\mathcal{R}G_d(Z)$ in terms of accessible geometric features of Z . The rigidity constant $\mathcal{R}G_d(Z)$ is closely related to some of the central notions in the modern smooth extension theory (see [3, 6, 7] and references therein). We give more comments on this connection in Section 4.

The following one-dimensional example illustrates some patterns of smooth rigidity. Start with a basic property of polynomials: a nonzero univariate polynomial $P(x)$ of degree d can have at

most d real zeros. Here is the corresponding rigidity result (well-known in various forms). It bounds from below the $(d + 1)$ -st derivative of a function f , which violates this property:

Proposition 2.2. *For each $(d + 1)$ -smooth function $f(x)$ on $[-1, 1]$, with $\max_{[-1, 1]} |f(x)| \geq 1$ and with $d + 1$ or more distinct zeroes on $[-1, 1]$, we have*

$$\max_{[-1, 1]} |f^{(d+1)}(x)| \geq \frac{(d+1)!}{2^{d+1}}.$$

A short proof can be given using the Lagrange remainder formula in the polynomial interpolation of f on its zeroes, or (more or less equivalently) via divided finite differences (see [17]).

In terms of the rigidity constant $\mathcal{R}G_d(Z)$ Proposition 2.2 implies

Proposition 2.3. *For any $Z \subset B^1$ we have $\mathcal{R}G_d(Z) \geq \frac{(d+1)!}{2^{d+1}}$, if Z consists of at least $d + 1$ different points, and $\mathcal{R}G_d(Z) = 0$ if Z consists of at most d different points.*

The corresponding “near-polynomiality” result is the following:

Corollary 2.4. *Any $(d + 1)$ -smooth function $f(x)$ on $[-1, 1]$, with*

$$\max_{[-1, 1]} |f(x)| \geq 1, \quad \max_{[-1, 1]} |f^{(d+1)}(x)| < \frac{(d+1)!}{2^{d+1}}$$

has at most d zeroes in $[-1, 1]$.

In higher dimensions the powerful one-dimension tools such as Lagrange’s remainder formula, and divided finite differences, are not directly applicable. Still, Proposition 2.2 implies, via line sections, the following

Proposition 2.5. *For any $Z \subset B^n$ with a non-empty interior,*

$$\mathcal{R}G_d(Z) \geq \frac{(d+1)!}{2^{d+1}}.$$

Proof: Let $f \in U_d(Z)$. Fix a certain point x_1 with $|f(x_1)| = 1$, fix x_2 in the interior of Z , and let ℓ be the straight line through x_1, x_2 . The restriction \bar{f} of f to the intersection of ℓ with B^n has an entire interval of zeroes near x_2 , and it satisfies $M_0(\bar{f}) = 1$. Applying Proposition 2.2 to \bar{f} completes the proof. \square

Considering restrictions of f to the straight lines ℓ as above, and applying Proposition 2.2, we obtain in [14], in particular, the following “near-polynomiality” result:

Theorem 2.6. ([14]) *Let $f(x)$ be a smooth function on B^n , with $M_0(f) = 1$. If $M_d(f) \leq 2^{-d-1}$, for some $d \geq 1$, then the set of zeroes Y of f is contained in a countable union of smooth hypersurfaces, and the $(n - 1)$ -Hausdorff measure of Y is bounded by a constant, depending only on n and d .*

In turn, we immediately obtain a certain multi-variate rigidity inequality: if for some $d \geq 1$ the set of zeroes Z of a smooth function f on B^n of sup-norm 1 violates the restrictions of Theorem 2.6, then $M_d(f)$ is at least 2^{-d-1} .

However, the above approach to estimating $\mathcal{R}G_d(Z)$, based on restrictions of f to some straight lines ℓ , works only in situations, where “many” straight lines intersect Z at “many” points. Essentially, (via the integral-geometric interpretation) this is the case of sets $Z \subset B^n$, containing hypersurfaces of a big Hausdorff $(n - 1)$ -measure. Still, for many natural classes of candidate zero sets Z , this condition is not satisfied. For example, this is the case for Z being a finite set (unless, by a rare coincidence, many points of Z lie on the same straight line). The

same remains true if we replace each point of such a finite set Z with a small hypersurface around it.

The main goal of [17] was to develop a pure multi-dimensional approach to smooth rigidity, based on polynomial Remez-type inequalities (which compare the maxima of a polynomial on the closed unit ball, and on its closed subset Z). Loosely speaking, *one of the main results of [17] was that the d -rigidity of a set Z is approximately the reciprocal d -Remez constant of Z .*

3. REMEZ CONSTANT

Let us recall the definition and some basic properties of the Remez (or Lebesgue, or norming) constant. See [4, 2, 5, 15] for more details and references.

Definition 3.1. *For a set $Z \subset B^n \subset \mathbb{R}^n$ and for each $d \in \mathbb{N}$ the Remez constant $\mathcal{R}_d(Z)$ is the minimal K for which the inequality*

$$\sup_{B^n} |P| \leq K \sup_Z |P|$$

is valid for any real polynomial $P(x) = P(x_1, \dots, x_n)$ of degree d .

Clearly, we always have $\mathcal{R}_d(Z) \geq 1$. For some Z the Remez constant $\mathcal{R}_d(Z)$ may be equal to ∞ . In fact, $\mathcal{R}_d(Z)$ is infinite if and only if Z is contained in the set of zeroes

$$Y_P = \{x \in \mathbb{R}^n, \mid P(x) = 0\}$$

of a certain polynomial P of degree d . We call sets Z with finite $\mathcal{R}_d(Z)$ d -norming. We use also the reciprocal Remez constant $\hat{\mathcal{R}}_d(Z) := \frac{1}{\mathcal{R}_d(Z)}$.

3.1. Rigidity constant via Remez constant. An important initial observation, connecting the Remez and rigidity constants is:

Lemma 3.2. ([17]) $\mathcal{R}G_d(Z) = 0$ if and only if $\hat{\mathcal{R}}_d(Z) = 0$.

See [17] for the proof. The following is one of the main results of [17]. It is based, in particular, on [16]:

Theorem 3.3. ([17]) *For any $Z \subset B^n$, we have $\frac{(d+1)!}{2} \hat{\mathcal{R}}_d(Z) \leq \mathcal{R}G_d(Z)$, or*

$$\mathcal{R}G_d(Z) \cdot \mathcal{R}_d(Z) \geq \frac{(d+1)!}{2}.$$

This lower bound is valid for any Z , and it is sharp, up to constants (depending only on n and d , and on the separation between the point), for finite sets, as Theorem 3.4 below shows. However, we cannot expect *an upper bound* of the form

$$(3.1) \quad \mathcal{R}G_d(Z) \leq C(n, d) \hat{\mathcal{R}}_d(Z),$$

for some constant $C(n, d)$ depending only on n and d , to be valid in general: indeed, by Proposition 2.5, for any $Z \subset B^n$ with a non-empty interior, $\mathcal{R}G_d(Z) \geq \frac{(d+1)!}{2^{d+1}}$. On the other hand, sets Z with a non-empty interior may have arbitrarily small Remez constant $\hat{\mathcal{R}}_d(Z)$. For example, let P be a polynomial of degree d with $M_0(P) = 1$, and for some $\eta > 0$ let Z be the η -sublevel set of P , i.e. $Z = \{z \in B^n, \mid P(z) \leq \eta\}$. Clearly, Z has a non-empty interior, and we have $\hat{\mathcal{R}}_d(Z) \leq \eta$.

Still, for some important types of sets Z an upper bound for the rigidity through the Remez constant holds. In [17] we prove it for finite sets Z with a controlled minimal distance between the points:

Theorem 3.4. ([17]) *Let $Z \subset B^n$ be a finite set, and let ρ be the minimal distance between the points of Z . Then*

$$\frac{(d+1)!}{2} \hat{\mathcal{R}}_d(Z) \leq \mathcal{R}G_d(Z) \leq \frac{C(n,d)}{\rho^{d+1}} \hat{\mathcal{R}}_d(Z).$$

This theorem can be considered as a generalization of Proposition 2.3 to higher dimensions.

In dimensions 2 and higher we have finite sets Z with positive but arbitrarily small $\hat{\mathcal{R}}_d(Z)$, and with ρ , uniformly bounded from below. For such sets the upper bound of Theorem 3.4 is meaningful. One of the simplest examples is a plane triangle Z_h , defined as

$$Z_h = \left\{ \left(-\frac{1}{2}, 0\right), (0, h), \left(\frac{1}{2}, 0\right) \right\}.$$

Easy computation shows that $\hat{\mathcal{R}}_1(Z_h) = \frac{h}{2}$.

4. SMOOTH RIGIDITY AND WHITNEY EXTENSIONS

In this section we discuss, quite informally, some very important connections between the smooth rigidity and Whitney smooth extension (see [11, 12, 13, 3, 6, 7]). In fact, our Definition 2.1 of the d -rigidity $\mathcal{R}G_d(Z)$ is a special case of one of the main notions in the Whitney smooth extension theory. Indeed, in the Whitney C^m -extension problem we consider a closed subset $Z \subset B^n$, and a function \tilde{f} on Z . The question is whether \tilde{f} is extendable to a C^m -smooth function f on B^n , and if so, what is the minimal C^m -norm of the smooth extensions f of \tilde{f} to B^n . In our Definition 2.1 we just assume that $\tilde{f} \equiv 0$ on Z , add the requirement $M_0(f) = 1$, and ask for the minimal $M_d(f)$ of the extensions f .

In dimension $n = 1$ the Whitney extension theorem of [12] provides the complete and explicit answer to the C^m -extension question: it is possible if and only if all the divided finite differences of \tilde{f} on the subsets of Z of cardinality at most $m+1$ are uniformly bounded. The minimal C^m -norm of the extensions is also estimated through these finite differences.

As a consequence we can produce an explicit expression for the d -th rigidity constant $\mathcal{R}G_d(Z)$. Consider all the subsets $\tilde{Z} = \{z_0, z_1, \dots, z_{d+1}\}$, with $z_0 \in [-1, 1] \setminus Z$ and $z_1, \dots, z_{d+1} \in Z$, and let $\Delta_{d+1}(\tilde{Z})$ denote the $d+1$ -st divided finite difference on \tilde{Z} of the function

$$y_0 = 1, y_1 = \dots = y_{d+1} = 0.$$

Then the d -th rigidity constant $\mathcal{R}G_d(Z)$ can be estimated as the infimum over all the subsets \tilde{Z} as above of $\Delta_{d+1}(\tilde{Z})$.

There is a fundamental difficulty in extending one-dimensional polynomial interpolation and smooth extension results to higher dimensions. This difficulty manifests itself in many ways, but for our purposes it can be shortly summarized as follows: *in dimensions greater than one there are no canonical divided finite differences*. Even the following most basic question, directly suggested by Whitney's one-dimensional results, was open for many years:

In order to check a C^m -extensibility of \tilde{f} on $Z \subset B^n$, and to estimate a minimal C^m -norm of the extension, is it enough to check only subsets of Z of a fixed cardinality $N = N(n, m)$?

A remarkable progress was achieved in the multi-dimensional Whitney extension problem in the last two decades (see [3, 6, 7] and references therein). In particular, the above question was ultimately answered positively in [6]. As a result, we can provide an explicit expression for the d -th rigidity constant $\mathcal{R}G_d(Z)$ through the rigidity constants $\mathcal{R}G_d(\tilde{Z})$, where \tilde{Z} runs over all the finite subsets of Z of cardinality $N = N(n, d)$. However, in a strict contrast with the one-dimensional case, the rigidity constants $\mathcal{R}G_d(\tilde{Z})$ for finite sets \tilde{Z} of cardinality N are not given anymore by a simple formula.

Accordingly, we come to an important problem of providing some “computable” bounds for $\mathcal{R}G_d(Z)$ for *finite sets* Z in terms of their geometry. Some results in this direction were obtained in [17]. In particular, besides the results in Sections 2 and 3 above, we have the following “density” lower bound for the rigidity $\mathcal{R}G_d(Z)$:

Theorem 4.1. ([17]) *Let $Z \subset B^n$ be a finite set, and let ρ be the minimal distance between the points of Z . Assume that the cardinality $M = |Z|$ satisfies $M > (4d)^n (\frac{1}{\rho})^{n-1}$. Then*

$$0 < \frac{(d + 1)!}{2} \left(\frac{M\rho^n - (4d)^n \rho}{4n} \right)^d \leq \mathcal{R}G_d(Z).$$

Informally, if in resolution ρ the set Z “looks more massive than an algebraic hypersurface of degree d ”, then its d -th rigidity constant is positive.

Some additional results, in terms of the asymptotic behavior of the covering number of \bar{Z} , were obtained in [18]. We plan to present separately our results on connections between smooth rigidity and Whitney extensions.

5. WHAT IS THE DEGREE OF A SMOOTH HYPERSURFACE (AFTER A. LERARIO AND M. STECCONI, [8])

In this section we continue the review part of the present paper. We discuss, from the point of view of smooth rigidity, an important recent progress in Global Singularity Theory, achieved in [8]. Informally, the results of [8] claim that, under appropriate transversality conditions, singular loci of smooth functions are isotopic to the corresponding singular loci of algebraic polynomials of explicitly bounded degree. It is shown, in particular, that if f as above vanishes *transversally* on a smooth manifold $Z \subset B^n$, and if high-order derivatives of f are small enough, then Z is smoothly isotopic to a component of a zero set of a polynomial P of a “small” degree. We expect that the results of [8] will lead to some important “rigidity via topology” inequalities for singular loci of smooth functions.

Let’s quote [8] and state one of its main results more accurately. For a smooth $f : B^n \rightarrow \mathbb{R}$ with a compact and smooth zero set $Y(f)$ the following problem is considered: what is the smallest degree of a polynomial P whose zero set $Y(P)$ is diffeomorphic to $Y(f)$? More generally, for W a closed semialgebraic subset of the jet space $J^r(B^n, \mathbb{R}^q)$ we consider the (type W) singularity $Y_W(f)$, defined as

$$Y_W(f) = (j^r f)^{-1}(W),$$

where $j^r f : B^n \rightarrow J^r(B^n, \mathbb{R}^q)$ is the r -jet extension of f . Certainly, some transversality assumptions are necessary in order to study the topology of the singular sets $Y_W(f)$. Otherwise, even in the simplest case of the zero level hypersurfaces of f any closed subset of B^n may appear as $Y(f)$ for a certain C^∞ function f .

Define the discriminant set Δ_W as the set of all $f \in C^{r+1}$, whose r -jet extension $j^r f : B^n \rightarrow J^r(B^n, \mathbb{R}^q)$ is not transversal to W . Quantitatively, the transversality of f to W is measured by the distance $\delta_W(f)$ from f to Δ_W in the space of C^{r+1} -smooth functions.

Theorem 5.1. ([8], Theorem 1) *Let $W \subset J^r(B^n, \mathbb{R}^q)$ be closed semialgebraic. For every $f \in C^{r+2}(B^n, \mathbb{R}^q)$ with $j^r f$ transversal to W there exists a polynomial map $P = (P_1, \dots, P_q)$ with*

$$\deg(P_i) \leq C(r) \max \left\{ r + 1, \frac{\|f\|_{C^{r+2}}}{\delta_W(f)} \right\},$$

such that the (type W) singularity $Y_W(f)$ is smoothly isotopic to $Y_W(P)$.

This result allows the authors to immediately bound the topological complexity of $Y_W(f)$ (see [8], Corollary 2).

From the smooth rigidity point of view, Theorem 5.1 immediately implies the following result, where we put $\bar{C}(r) = \frac{1}{C(r)}$:

Theorem 5.2. *For f as above, and for a certain closed semialgebraic $W \subset J^r(B^n, \mathbb{R}^q)$, assume that the (type W) singularity $Y_W(f)$ is not smoothly isotopic to $Y_W(P)$ for any polynomial map P of a given degree $d \geq r + 1$. Then*

$$\|f\|_{C^{r+2}} \geq \bar{C}(r) \cdot d \cdot \delta_W(f).$$

Of course, the condition of $Y_W(f)$ not being smoothly isotopic to $Y_W(P)$ for any polynomial map P of degree $d \geq r + 1$ can be replaced by stronger conditions in terms of topological complexity of $Y_W(P)$ (for example, in terms of the sum of its Betti numbers).

If we take the degree d as the “measure of complexity” of polynomials, then Theorem 5.2 implies the *linear grows of the norm* $\|f\|_{C^{r+2}}$ with d , at least. It would be interesting to make this very initial observation more rigorous.

We believe that an explicit and detailed study, in the above directions, of the *specific* singularity types W is well justified. As an example, in Section 7 below we provide a strongly simplified (and restricted only to the zero hypersurfaces $Y(f)$) version of Theorem 5.1. Our goals there are to get explicit bounds, to estimate separately the $(r + 1)$ -st derivative of f , and to provide simple geometric arguments in the proof of this specific case.

6. “TOPOLOGICAL” REMEZ-TYPE INEQUALITY

Starting with this section, we show the new results of the present paper. Remez-type inequalities provide an upper bound for $\mathcal{R}_d(Z)$ in terms of various “computable” characteristics of Z . In particular, the classical multi-dimensional Remez inequality ([2], [10], see also [5]) uses the Lebesgue measure of Z . It reads as follows:

Theorem 6.1. *For any measurable $Z \subset B^n$ we have*

$$(6.1) \quad \mathcal{R}_d(Z) \leq T_d \left(\frac{1 + (1 - \lambda)^{\frac{1}{n}}}{1 - (1 - \lambda)^{\frac{1}{n}}} \right) \leq \left(\frac{4n}{\lambda} \right)^d.$$

Here $T_d(t) = \cos(d \arccos t)$ is the Chebyshev polynomial of degree d , and $\lambda = \frac{m_n(Z)}{m_n(B^n)}$, with m_n being the Lebesgue measure on \mathbb{R}^n . This inequality is sharp and for $n = 1$ it coincides with the classical Remez inequality of [10].

Some other examples of Remez-type inequalities, and a more detailed discussion can be found in [4, 5, 15]. In particular, it was shown in [15] that the Lebesgue measure can be replaced in Theorem 6.1 with a more sensible geometric invariant $\omega_{n,d}(Z)$, defined in terms of the covering numbers of Z . The invariant $\omega_{n,d}(Z)$ always satisfies $m_n(Z) \leq \omega_{n,d}(Z)$, so its substitution instead of the Lebesgue measure into (6.1) can only improve the result. The invariant $\omega_{n,d}(Z)$ allows us to distinguish between various discrete and even finite sets of different geometry and density.

However, there are natural classes of sets Z , for which Theorem 6.1, as well as its strengthening, where $m_n(Z)$ is replaced with $\omega_{n,d}(Z)$, do not work. Consider, for instance, smooth compact hypersurfaces $Z \subset B^n$. Their n -measure is zero, and if their $(n - 1)$ -area is small, then also $\omega_{n,d}(Z) = 0$. Still Z may be a d -norming set by “topological” reasons. Theorem 6.2 below is one of the main results of the present paper. It provides a Remez-type inequality for smooth compact hypersurfaces $Z \subset B^n$, in terms of the number of their connected components, and of

the n -volume of the *interiors of these components* (and not of the components themselves, as in Theorem 6.1).

We state this result in a slightly more general way: let U_j , $j = 1, \dots, N$, be nonintersecting compact connected domains in B^n with nonempty interiors, and let $\mu_j = m_n(U_j)$ be the n -measure of U_j . It is convenient to assume that $\mu_1 \geq \mu_2 \geq \dots \geq \mu_N$. Put $Z_j = \partial U_j$, to be the boundary of U_j , and let $Z = \cup_{j=1}^N Z_j$.

Define \bar{d} by $(\bar{d} - 1)^n + 1 \leq N < \bar{d}^n + 1$, and for each natural $d \leq \bar{d}$ put $j_d = (d - 1)^n + 1$.

Theorem 6.2. *With the above notations, for each $d \leq \bar{d}$ we have*

$$\mathcal{R}_d(Z) \leq \left(\frac{4n}{\mu_{j_d}} \right)^d.$$

Proof:

Let $d \leq \bar{d}$ be fixed, and let P be a polynomial of degree d with $M_0(P) = 1$. To prove Theorem 6.2 it is sufficient to show that $\max_Z |P| \geq \kappa_d := \left(\frac{\mu_{j_d}}{4n} \right)^d$. Thus we assume, in contrary, that $\max_Z |P| < \kappa_d$, and bring this assumption to a contradiction.

We have the following lemma:

Lemma 6.3. *For each U_j , $j = 1, \dots, N$, we have*

$$\max_{U_j} |P(x)| \geq \left(\frac{\mu_j}{4n} \right)^d.$$

Proof: By the classical multidimensional Remez inequality, given by Theorem 6.1 above, and applied to the set U_j , we have

$$1 = M_0(P) \leq \left(\frac{4n}{\mu_j} \right)^d \max_{U_j} |P(x)|, \quad \text{or} \quad \max_{U_j} |P(x)| \geq \left(\frac{\mu_j}{4n} \right)^d.$$

This completes the proof of Lemma 6.3. \square

Next we notice that if $\max_Z |P| < \kappa_d$, then for each $j = 1, \dots, j_d$, the polynomial P has a local maximum (or minimum) at a certain point \bar{x}_j in the interior of U_j . Indeed, in this case, by Lemma 6.3, the maximum of $|P|$ inside U_j satisfies

$$\max_{U_j} |P(x)| \geq \left(\frac{\mu_j}{4n} \right)^d \geq \left(\frac{\mu_{j_d}}{4n} \right)^d = \kappa_d,$$

while $\max_Z |P| < \kappa_d$. Hence the maximum of $|P|$ in U_j is strictly greater than its maximum on the boundary Z_j . In particular, the point \bar{x}_j , where this maximum is achieved, is a critical point of P , i.e. $\text{grad } P(\bar{x}_j) = 0$.

Consequently, if $\max_Z |P(x)| < \kappa_d$, then in the interior of each domain U_j , $j = 1, \dots, j_d$, there is a critical point of P , which is a local maximum or a local minimum of P . Performing a small perturbation of P we can assume that all the critical points of P are non-degenerate.

It remains to bound from above the maximal possible total number of non-degenerated maxima and minima of a polynomial P of degree d of n variables. By Bezout theorem, the total number of non-degenerated critical points of P , i.e. of the solutions of the system

$$\frac{\partial f}{\partial x_i} = 0, \quad i = 1, \dots, n,$$

cannot exceed $(d - 1)^n$. This contradiction completes the proof of Theorem 6.2. \square

Now, let Z be as in Theorem 6.2. Combining Theorems 6.2 and 3.3, we obtain a ‘‘topological’’ rigidity inequality:

Theorem 6.4. For each $d \leq \bar{d}$ we have

$$\mathcal{R}G_d(Z) \geq \frac{(d+1)!}{2} \left(\frac{\mu_{j_d}}{4n}\right)^d.$$

6.1. Some examples and remarks. An immediate corollary of Theorem 6.2 is the following:

Corollary 6.5. Let the degree d be given. Put $N = (d-1)^n + 1$, and let U_j , $j = 1, \dots, N$, be nonintersecting compact connected domains in B^n with nonempty interiors. As above, $Z_j = \partial U_j$ denote the boundaries of U_j , and $\mu_j = m_n(U_j)$ is the n -measure of U_j . Then the set $Z = \cup_{j=1}^N Z_j$ is d -norming, and

$$\mathcal{R}_d(Z) \leq \left(\frac{4n}{\mu_N}\right)^d.$$

In a special case of exactly one domain we have the following corollary of Theorem 6.2:

Corollary 6.6. Let Z be the boundary of the compact connected domain U in B^n , $n \geq 1$, with the n -volume of U being μ . Then we have

$$\mathcal{R}_1(Z) \leq \frac{4n}{\mu}.$$

Corollary 6.7. Let Z_1, Z_2 be the boundaries of the compact connected nonintersecting domains U_1, U_2 in B^n , $n \geq 1$, with the n -volumes of U_1, U_2 being $\mu_1 \geq \mu_2$. Then for $Z = Z_1 \cup Z_2$ we have

$$\mathcal{R}_2(Z) \leq \left(\frac{4n}{\mu_2}\right)^2.$$

Proof: For any $n \geq 1$ we have, by definition, $j_1 = 1$, $j_2 = 2$. Hence the result follows directly from Theorem 6.2. \square

Theorem 6.2 is sharp up to constants (depending on n and d) with respect to the volume μ_{j_d} and with respect to the required number N of the domains U_j . We give here in detail only the simplest example, for $n = 2$ and $d = 2$, i.e. we consider Corollary 6.7 instead of Theorem 6.2. Then we shortly discuss also the general case.

For a given $h > 0$ consider a polynomial $P_h(x, y) = h^2x^2 + y^2 - \frac{1}{4}h^2$. The zero set Y_h of $P_h(x, y)$ is the ellipse centered at the origin, with the semiaxes $\frac{1}{2}$ and $\frac{h}{2}$ in the directions Ox and Oy , respectively. Now in Corollary 6.7 we put U_1 to be the interior of the ellipse Y_h , and U_2 to be the interior of the rectangle

$$Q_h = \left\{ (x, y) \mid -\frac{1}{4} \leq x \leq \frac{1}{4}, \quad \frac{2h}{3} \leq y \leq \frac{3h}{4} \right\}.$$

We see immediatly that just one oval in Z is not enough to make Z a 2-norming set. Indeed, Z_1 is the zero set the polynomial P_h of degree 2. Next, the maximum of $|P_h|$ on Q_h does not exceed h^2 while the maximum of $|P_h|$ on B^2 is at least $1 - \frac{1}{4}h^2$. We conclude that for $Z = Z_1 \cup Z_2$ we have

$$\mathcal{R}_2(Z) \geq \frac{1 - \frac{1}{4}h^2}{h^2} \geq \frac{1}{2h^2}$$

for small h . On the other hand, the smallest area of U_1, U_2 is of U_2 , and we have $\mu_2 = \frac{h}{48}$. Thus the bound of Corollary 6.7, for $n = 2$, takes the form

$$\mathcal{R}_2(Z) \leq \left(\frac{4n}{\mu_2}\right)^2 = \left(\frac{8 \cdot 48}{h}\right)^2 = \frac{147456}{h^2}.$$

Therefore, the power, with which the volume μ_2 enters the bound, is accurate, while the bound itself is sharp, up to a constant.

The requirement of Theorem 6.2 to have at least $N = (d - 1)^n + 1$ different disjoint domains U_j remains relatively sharp also in higher degrees and dimensions.

Consider the following polynomial $P(x_1, \dots, x_n)$ of degree nd :

$$P(x_1, \dots, x_n) = \prod_{i=1}^n Q(x_i), \quad Q(t) = (t - \eta_1)(t - \eta_2) \cdot \dots \cdot (t - \eta_d),$$

with η_1, \dots, η_d pairwise distinct numbers in the open interval $(-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})$. The zero set Y of the polynomial P is the union of all the shifted coordinate hyperplanes

$$x_i = \eta_j, \quad i = 1, \dots, n, \quad j = 1, \dots, d.$$

On each connected component of the complement $W = B^n \setminus Y$ the polynomial P preserves its sign, and it changes its sign as the argument crosses Y . The complement W contains, in particular, $(d - 1)^n$ adjoint cubes, on a half of them P being positive, and on a half negative. Now fix a small positive number ζ , which is a regular value of P , put $\bar{P} = P - \zeta$, and consider the sublevel set $V = \{x \in B^n, \bar{P}(x) \geq 0\}$. We take as $U_j, j = 1, \dots, \frac{1}{2}(d - 1)^n$ all the compact connected components of V , inside the cubes, where P was positive. The boundaries Z_j of U_j are smooth compact hypersurfaces, contained in the zero set \bar{Y} of \bar{P} . Therefore for $Z = \cup Z_j$ we have $\mathcal{R}_{nd}(Z) = \infty$. Replacing d by $\frac{d}{n}$ we produce an example of $\hat{N} = \frac{1}{2}(\frac{d}{n} - 1)^n$ disjoint connected domains \hat{U}_j for which $\hat{Z} = \cup \hat{Z}_j$ is not a d -norming set. We conclude that the required number $N = (d - 1)^n + 1$ of the domains U_j in Theorem 6.2 is sharp in the degree d , up to a constant depending only on the dimension n .

We expect that the power d , with which the volume μ_{j_d} enters the bound, is accurate. However, the above construction immediately produces examples with $\mathcal{R}_d(Z) \sim (\frac{1}{\mu_{j_d}})^{\frac{d}{n}}$ only. Indeed, we can consider the product of the polynomials $Q(x_i)$ as above, with the roots of the first one down-scaled to the size h . Then all the domains U_j constructed belong to the strip $|x_1| \leq h$. Repeating verbally the construction of the example after Corollary 6.7, we obtain $\mathcal{R}_d(Z) \sim (\frac{1}{h})^{\frac{d}{n}}$, while $\mu_{j_d} \sim h$.

Notice that for $d = 1, 2$ and for any n , the bound $(d - 1)^n + 1 = 1, 2$, on the number of the connected components of Z , is sharp. For $d \geq 3$, in many cases this bound can be improved, since by topological reasons, other critical points, beyond maxima and minima, must appear. Still, we are aware only of some partial estimates of the possible number of minima and maxima of real polynomials. In particular, one can show that the number of extrema of P does not exceed $\frac{1}{2}d^n + O(d^{n-1})$. For $n=3$, in [1] there are examples with at least $\frac{13}{36}d^n$ extrema. Also the lower bound $(\frac{2^{n-1}}{n!})d^n$ is known. In a recent paper [9] it is shown that (on the real sphere) homogeneous polynomials can have all the (a priori complex) solutions of the critical point equations to be real. (The author thanks E. Shustin and the referee for providing some references).

We consider an accurate estimate of the required number of the connected components of Z in Theorem 6.2 as an interesting question in real algebraic geometry, closely related not only to bounding the number of the extrema of P , but also to the topology and mutual position of its ovals.

Returning to the product polynomial $P(x_1, \dots, x_n) = \prod_{i=1}^n Q(x_i)$, constructed above, we notice that P can be considered as a ‘‘poly-degree d ’’ polynomial, with respect to the appropriate Newton diagram. Many results of real algebraic geometry can be extended to such polynomials, as well as some Remez-type inequalities. We expect that the connection between these two topics, provided by Theorem 6.2, remain valid also for polynomials with a prescribed Newton diagram (and not only of a given degree).

7. TOPOLOGY OF TRANSVERSAL LEVEL SETS

In this section we illustrate in more detail the results of [8] and their consequences for smooth rigidity. For this purpose, and in order to provide as explicit geometric arguments as possible, we give below a direct proof of one very special case of the general results of [8]. On this base we provide also the corresponding rigidity inequality.

Let f be a C^{d+1} -smooth function on B^n , with $M_0(f) = 1$. For a given $\gamma > 0$, a real number c is called a γ -regular value of f , if for each $x \in B^n$ with $f(x) = c$, we have $\|\text{grad } f(x)\| \geq \gamma$. In this case the level set

$$Y_c(f) = \{x \in B^n, f(x) = c\}$$

is a regular compact manifold of dimension $n - 1$.

To simplify the presentation, and to avoid the boundary effects, we assume that f does not vanish out of the concentric ball $B_{\frac{1}{2}}^n$. Next we assume that for a certain γ , $0 < \gamma < 1$, zero is a γ -regular value of f . In the notations of [8], for W consisting of the jets with the zero value, γ is, essentially, the distance $\delta_W(f)$ of f to the discriminant set Δ_W .

Thus the zero level set $Y_0(f) = \{x \in B^n, f(x) = 0\}$ is a compact smooth hypersurface in $B_{\frac{1}{2}}^n$. Let V_i , $i = 1, \dots, q$, denote the connected components of $Y_0(f)$.

We put $T = T(\gamma) = \min \{1, \frac{d\gamma^2}{4C_3}\}$, where the constant $C_3 = C_3(n, d)$ is defined below. Finally, let $P = P_d(f)$ be the Taylor polynomial at the origin of degree d of f , and let $Y_0(P)$ be its zero set. The following result is (essentially) a very special case of Theorem 1.1 of [8] (see also Section 5 above):

Theorem 7.1. *If $M_{d+1}(f) \leq T$ then the smooth hypersurface $Y_0(f) = \cup_{i=1}^q V_i$ is smoothly isotopic to a certain union $W = \cup_{i=1}^q W_i$ of the smooth connected components W_i of the algebraic hypersurface $Y_0(P)$.*

Proof: In order to avoid complicated expressions, we use below constants $C_q(d, n)$, depending only on d and n , not specifying some of them explicitly.

The following lemma provides a bound for the norms of the intermediate derivatives $M_q(f)$ of a smooth f through $M_0(f)$ and $M_{d+1}(f)$.

Lemma 7.2. *Let f be a C^{d+1} -smooth function on B^n . Then for $k = 1, 2, \dots, d$ we have*

$$M_k(f) \leq C_1(n, d)M_0(f) + C_2(n, d)M_{d+1}(f).$$

Proof: Let $P = P_d(f)$ be the Taylor polynomial at the origin of degree d of f . By Taylor's formula we have for $x \in B^n$

$$|f(x) - P(x)| \leq \frac{1}{(d+1)!}M_{d+1}(f).$$

We conclude that $M_0(P) \leq M_0(f) + \frac{1}{(d+1)!}M_{d+1}(f)$. Next we use the equivalence of all the norms on the finite-dimensional space of polynomials of degree d , and obtain, for $k = 1, 2, \dots, d$,

$$M_k(P) \leq \bar{C}(n, d)M_0(P) \leq \bar{C}(n, d)[M_0(f) + \frac{1}{(d+1)!}M_{d+1}(f)].$$

Finally, once more using Taylor's formula, we get for $k = 1, 2, \dots, d$,

$$M_k(f) \leq M_k(P) + \frac{1}{(d-k+1)!}M_{d+1}(f) \leq C_1(n, d)M_0(f) + C_2(n, d)M_{d+1}(f),$$

where $C_1(n, d) = \bar{C}(n, d)$, $C_2(n, d) = \bar{C}(n, d)\frac{1}{(d+1)!} + \frac{1}{(d-k+1)!}$. This completes the proof of Lemma 7.2. \square

In particular, under our initial assumptions that $M_0(f) = 1$ and $M_{d+1}(f) \leq T \leq 1$ we have

$$M_2(f) \leq C_1(n, d) + C_2(n, d) := C_3(n, d).$$

Put $\delta = \frac{\gamma}{3C_3}$. Then, in a δ -neighborhood U_δ of $Y_0(f)$ we have $\|\text{grad } f(x)\| \geq \frac{1}{2}\gamma$.

Next we consider a vectorfield $v(x)$ in U_δ defined by $v(x) = \frac{\text{grad } f(x)}{\|\text{grad } f(x)\|^2}$. For $x \in U_\delta$ we have $\|v(x)\| \leq \frac{2}{\gamma}$.

The derivative of f in the direction of $v(x)$ satisfies the identity

$$\frac{df}{dv}(x) = \langle v(x), \text{grad } f(x) \rangle = \frac{\langle \text{grad } f(x), \text{grad } f(x) \rangle}{\|\text{grad } f(x)\|^2} = 1.$$

For each $y \in Y_0(f)$ denote by $\zeta(y, t)$ the trajectory of the vectorfield $v(x)$, satisfying $\zeta(y, 0) = y$. Since for $x \in U_\delta$ we have $\|v(x)\| \leq \frac{2}{\gamma}$, for each t with $|t| \leq \frac{\delta\gamma}{2} = \frac{\gamma^2}{2C_3} := \eta$ the trajectory $\zeta(y, t)$ remains in U_δ , and hence it is well-defined.

Now we consider the “normal bundle” mapping $\Psi : G \rightarrow U_\delta$ of the product $G = Y_0(f) \times [-\eta, \eta]$ into U_δ , defined by

$$\Psi(y, t) = \zeta(y, t).$$

By the construction, Ψ satisfies $f(\Psi(y, t)) = t$. By the uniqueness and dependence on the initial data of the trajectories $\zeta(y, t)$, the mapping Ψ provides a diffeomorphism of G with its image, which is the level strip $Q_\eta = \{x \in U_\delta, |f(x)| \leq \eta\}$.

Let us return now to the Taylor polynomial P of degree d of f at the origin. By the remainder formula, and since we assume that $M_{d+1}(f) \leq T$, we have $M_0(f - P) \leq \frac{T}{(d+1)!} \leq \frac{1}{2}\eta$. Also by the Taylor formula we have $M_1(f - P) \leq \frac{T}{d!} \leq \frac{\gamma^2}{4C_3}$. We conclude that for $t \in [-\eta, \eta]$, and for each $y \in Y_0(f)$, along the trajectory $\zeta(y, t)$ we have $|\frac{dP}{dt} - 1| \leq \frac{1}{2}$.

Therefore, for each $y \in Y_0(f)$, along the trajectory $\zeta(y, t)$ the polynomial $P(\zeta(y, t))$ has exactly one simple zero at a certain $t(y) \in [-\eta, \eta]$. By the implicit function theorem, the function $t(y)$ is C^d -smooth.

Summarizing, we conclude that the part Ω of the zero set $Y_0(P)$ of P , which is contained in the level strip Q_η , is given in the coordinates y, t on Q_η as the graph of the smooth function $t(y)$ on $Y_0(f)$. Hence $\Omega = \cup_{i=1}^q \Omega_i$, where each Ω_i is the graph $t(y)$ on V_i . We immediately conclude also that Ω_i is diffeomorphic to V_i , $i = 1, \dots, q$. But in fact, the formula $t_\tau(y) = \tau t(y)$, $\tau \in [0, 1]$, provides a smooth isotopy between Ω and $Y_0(f)$. This completes the proof of Theorem 7.1. \square

The corresponding rigidity statement is

Theorem 7.3. *Let f be as above, with 0 a γ -regular value of f . If $Y_0(f)$ is not smoothly isotopic to a certain union $W = \cup_{i=1}^q W_i$ of smooth connected components W_i of an algebraic hypersurface $Y_0(P)$, with P a polynomial of degree d , then*

$$M_{d+1}(f) \geq T = T(\gamma) = \min \left\{ 1, \frac{d!\gamma^2}{4C_3} \right\}.$$

Remark 7.4. “No isotopy” condition of Theorem 7.3 can be weakened in many forms, in particular, in terms of the Betti numbers of the components of $Y_0(f)$, of their mutual position, etc.

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