Journal *of* **Singularities**
Volume 25 (2022), 268-298

Proc. of 16th International Workshop on Singularities, São Carlos, 2020
DOI: 10.5427/jsing.2022.25l

THE INTEGRAL MONODROMY OF THE CYCLE TYPE SINGULARITIES

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Abstract. The middle homology of the Milnor fiber of a quasihomogeneous polynomial with an isolated singularity is a $\mathbb{Z}$-lattice and comes equipped with an automorphism of finite order, the integral monodromy. Orlik (1972) made a precise conjecture, which would determine this monodromy in terms of the weights of the polynomial. Here we prove this conjecture for the cycle type singularities. A paper of Cooper (1982) with the same aim contained two mistakes. Still it is very useful. We build on it and correct the mistakes. We give additional algebraic and combinatorial results.

1. Introduction and main result

The main objects of the paper are a quasihomogeneous singularity, the monodromy on its Milnor lattice, Orlik's conjecture for this monodromy, and our proof in the case of a cycle type singularity. In order to make precise statements, we start with some algebraic definitions.

Definition 1.1. (a) Start with a product $p \in \mathbb{Z}[t]$ of cyclotomic polynomials which has only simple zeros. The *Orlik block* $\mathrm{Or}(p)$ is a pair (H, h) where H is a $\mathbb{Z}$-lattice of rank $\deg(p)$ and $h : H \to H$ is an automorphism of finite order with characteristic polynomial p such that an element $a_0 \in H$ with

$$H = \bigoplus_{j=0}^{\deg(p)-1} \mathbb{Z} \cdot h^j(a_0). \tag{1.1}$$

exists. Such an element is called a *generating element*. The Orlik block $\mathrm{Or}(p)$ is up to isomorphism uniquely determined by p, which justifies the notion $\mathrm{Or}(p)$.

(b) Consider a pair (H, h) where H is a $\mathbb{Z}$-lattice and $h : H \to H$ is an automorphism of finite order. It admits a *decomposition into Orlik blocks* if it is isomorphic to a direct sum of Orlik blocks.

(c) Consider a pair (H, h) where H is a $\mathbb{Z}$-lattice and $h : H \to H$ is an automorphism of finite order. Then the characteristic polynomial $p_{H,h}$ of h is a product of cyclotomic polynomials. It has a unique decomposition $p_{H,h} = \prod_{i=1}^{l} p_i$ with $p_l | p_{l-1} | ... | p_2 | p_1$ and $p_l \neq 1$ and all p_i unitary and such that p_1 has only simple zeros. The pair (H, h) admits a *standard decomposition into Orlik blocks*, if an isomorphism $(H, h) \cong \bigoplus_{i=1}^{l} \mathrm{Or}(p_i)$ exists.

A polynomial $f \in \mathbb{C}[x_1, ..., x_n]$ is called quasihomogeneous if for some weight system $(w_1, ..., w_n)$ with $w_i \in (0, 1) \cap \mathbb{Q}$ each monomial in f has weighted degree 1. It is called an isolated quasihomogeneous singularity if it is quasihomogeneous and the functions $\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}$ vanish simultaneously only at $0 \in \mathbb{C}^n$. Then the Milnor lattice $H_{Mil} := H_{n-1}^{(red)}(f^{-1}(1), \mathbb{Z})$ (here

2010 *Mathematics Subject Classification.* 55U15, 55T05, 58K10, 32S50.

Key words and phrases. Integral monodromy, cycle type singularity, Orlik's conjecture, spectral sequence.

This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 242588615.

$H_{n-1}^{(red)}$ means the reduced homology in the case $n = 1$ and the usual homology in the cases $n \geq 2$) is a $\mathbb{Z}$-lattice of some finite rank $\mu \in \mathbb{N}$, which is called the *Milnor number* [Mi68]. It comes equipped with an automorphism h_{Mil} of finite order, the *monodromy*.

Orlik conjectured the following.

Conjecture 1.2. *(Orlik's conjecture* [Or72, Conjecture 3.1]*) For any isolated quasihomogeneous singularity, the pair* (H_{Mil}, h_{Mil}) *admits a standard decomposition into Orlik blocks.*

Here we will prove this conjecture for the cycle type singularities. A *cycle type singularity* is a polynomial $f \in \mathbb{C}[x_1, ..., x_n]$ in $n \geq 2$ variables of the following shape,

$$f = x_1^{a_1} x_2 + x_2^{a_2} x_3 + ... + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1 \quad \text{where } a_1, ..., a_n \in \mathbb{N}, \tag{1.2}$$

($\mathbb{N} = \{1, 2, 3, ...\}$) and which is an isolated quasihomogeneous singularity. By Lemma 3.4 in [HK12], a polynomial f as in (1.2) is an isolated quasihomogeneous singularity if and only if n is odd or (1.3) holds,

$$n \text{ is even and } \begin{cases} \text{either } a_1 = ... = a_n = 1 \text{ or} \\ a_j \neq 1 \text{ for some even } j \text{ and for some odd } j. \end{cases} \tag{1.3}$$

Define $d := \prod_{j=1}^n a_j - (-1)^n$. Then (see e.g. Lemma 4.1 in [HZ19])

$$\mu = \prod_{j=1}^n a_j = d + (-1)^n,$$

and there are natural numbers $v_1, ..., v_n$ (which are given in (5.1)) such that

$$(w_1, ..., w_n) = \Big(\frac{v_1}{d}, ..., \frac{v_n}{d}\Big)$$

is the unique weight system for which f is quasihomogeneous of weighted degree 1. They satisfy also $\gcd(d, v_1) = ... = \gcd(d, v_n)$. Define $b := d/\gcd(d, v_1) \in \mathbb{N}$. An easy calculation which builds on the formula in [MO70] for the characteristic polynomial in terms of $(w_1, ..., w_n)$ (see e.g. Lemma 4.1 in [HZ19]) shows that the characteristic polynomial of h_{Mil} on H_{Mil} is

$$p_{H_{Mil}, h_{Mil}} = (t^b - 1)^{\gcd(d, v_1)} \cdot (t - 1)^{(-1)^n}. \tag{1.4}$$

Therefore Orlik's conjecture says here the following.

Theorem 1.3. *For a cycle type singularity as above,*

$$(H_{Mil}, h_{Mil}) \cong \begin{cases} (\gcd(d, v_1) - 1)\mathrm{Or}(t^b - 1) \oplus \mathrm{Or}(\frac{t^b - 1}{t - 1}) & \text{if } n \text{ is odd,} \\ \gcd(d, v_1)\mathrm{Or}(t^b - 1) \oplus \mathrm{Or}(t - 1) & \text{if } n \text{ is even.} \end{cases}$$

Our proof in section 5 builds on Cooper's work [Co82]. By [Mi68], the set

$$\overline{F_0} := f^{-1}(\mathbb{R}_{\geq 0}) \cap S^{2n-1} \subset S^{2n-1} \subset \mathbb{C}^n$$

is diffeomorphic to the Milnor fiber $f^{-1}(1)$. Cooper studied for the cycle type singularities a beautiful subset $G \subset \overline{F_0}$, which is a probably a deformation retract of the Milnor fiber (Cooper's Lemma 3 is slightly weaker). He considered certain *cells* from which this set G is built up and which allow to filter G by a sequence of subsets $G = G_n \supset G_{n-1} \supset ... \supset G_1 \supset G_0 = \emptyset$. Finally, he studied a spectral sequence which comes from this filtration.

Cooper claimed to have proved Orlik's conjecture for the cycle type singularities. But his paper contains two serious mistakes. The second one leads in the case of even n to the wrong claim

$$(H_{Mil}, h_{Mil}) \cong (\gcd(d, v_1) - 1)\mathrm{Or}(t^b - 1) \oplus \mathrm{Or}\Big(\frac{t^b - 1}{t - 1}\Big) \oplus 2\mathrm{Or}(t - 1).$$

The right-hand side is a decomposition into Orlik blocks, but not a standard decomposition.

The history of Orlik's conjecture is as follows. Michel and Weber claimed in the introduction of [MW86] to have a proof of Orlik's conjecture in the case $n = 2$. We trust this claim. Hertling [He92] proved Orlik's conjecture for some cases with $n = 3$ by explicit calculations using Coxeter-Dynkin diagrams. A chain type singularity is a polynomial $f \in \mathbb{C}[x_1, ..., x_n]$ in $n \geq 1$ variables of the shape $x_1^{a_1+1} + x_1 x_2^{a_2} + ... + x_{n-1} x_n^{a_n}$ where $a_1, ..., a_n \in \mathbb{N}$. It is an isolated quasihomogeneous singularity. Orlik and Randell [OR77, Theorem (2.11)] proved for it that (H_{Mil}, h^{μ}_{Mil}) is a single Orlik block. Together with two algebraic results in [HM20], this implies Orlik's conjecture for the chain type singularities [HM20, Theorem 1.3 (a)]. Two other algebraic results in [HM20] imply Orlik's conjecture for the Thom-Sebastiani sum $f + g$ of two isolated quasihomogeneous singularities f and g which both satisfy Orlik's conjecture [HM20, Theorem 1.3 (c)]. This relative result can be combined with the facts that Orlik's conjecture holds for the chain type singularities and the cycle type singularities (main result of this paper). We obtain Orlik's conjecture for all iterated Thom-Sebastiani sums of chain type singularities and cycle type singularities [HM20, Theorem 1.3 (d)]. This surpasses all known cases. These singularities are precisely the *invertible polynomials*. They form an important subfamily of all isolated quasihomogeneous singularities.

We will discuss the relation of this paper to Cooper's work and the two mistakes in the Remarks 5.1. His work is the basis for the sections 3 and 4 below. Section 2 gives new algebraic results. Section 3 gives the set G, its cells and an inductive construction of cycles of which only the beginning is in [Co82]. Section 4 makes good use of the spectral sequence which Cooper considered and determines $H_{n-1}(G, \mathbb{Z})$. A combination of the results of the sections 2, 3 and 4 and a discussion of the monodromy proves Theorem 1.3 in section 5.

2. Algebraic results

The following two lemmata 2.3 and 2.4 are elementary. They will be used in the proof of Orlik's conjecture for cycle type singularities with an even number of variables. They say something about the $\mathbb{Z}$-lattices $H^{(d,c)}$ with automorphisms $h^{(d,c)}$ of finite order, which are defined in Definition 2.1.

Definition 2.1. Let $d \in \mathbb{N}$ and $c \in \mathbb{Z}$. Define the pair $\mathrm{Lo}^{(d,c)} = (H^{(d,c)}, h^{(d,c)})$ as follows. $H^{(d,c)} = \mathbb{Z}\cdot\gamma \oplus \bigoplus_{j=1}^{d-1} \mathbb{Z}\cdot\delta_j$ is a $\mathbb{Z}$-lattice of rank d. Define additionally $\delta_d := c\cdot\gamma - \sum_{j=1}^{d-1} \delta_j \in H^{(d,c)}$. Then $h^{(d,c)} : H \to H$ is the automorphism of finite order d which is defined by

$$h^{(d,c)} : \ \gamma \mapsto \gamma, \quad \delta_d \mapsto \delta_1, \quad \delta_j \mapsto \delta_{j+1} \text{ for } j \in \{1, ..., d-1\}. \tag{2.1}$$

Remark 2.2. The characteristic polynomial of $h^{(d,c)}$ is $t^d - 1$. If $c \neq 0$, then

$$\sum_{j=1}^{d} \mathbb{Z} \cdot \delta_j = \bigoplus_{j=1}^{d} \mathbb{Z} \cdot \delta_j$$

is an $h^{(d,c)}$-invariant sublattice of index $|c|$ in $H^{(d,c)}$, and $(\bigoplus_{j=1}^{d} \mathbb{Z} \cdot \delta_j, h^{(d,c)}) \cong \mathrm{Or}(t^d - 1)$. Therefore $\mathrm{Lo}^{(d,1)} \cong \mathrm{Or}(t^d - 1)$. If $c = 0$ then the summands $\mathbb{Z} \cdot \gamma$ and $\bigoplus_{j=1}^{d-1} \mathbb{Z} \cdot \delta_j$ of $H^{(d,c)}$ are $h^{(d,c)}$-invariant with $(\mathbb{Z} \cdot \gamma, h^{(d,c)}) \cong \mathrm{Or}(t - 1)$ and $(\bigoplus_{j=1}^{d-1} \mathbb{Z} \cdot \delta_j, h^{(d,c)}) \cong \mathrm{Or}(\frac{t^d-1}{t-1})$.

Lemma 2.3. *Let $d, v \in \mathbb{N}$, $c \in \mathbb{Z}$ and $b := d/\gcd(d, v) \in \mathbb{N}$. Then*

$$(H^{(d,c)}, (h^{(d,c)})^v) \cong (\gcd(d, v) - 1)\mathrm{Or}(t^b - 1) \oplus \mathrm{Lo}^{(b,c)}. \tag{2.2}$$

Proof: Write $h := (h^{(d,c)})^v$. The elements $\delta_1, ..., \delta_d$ can be renumbered to elements $\widetilde{\delta}_1, ..., \widetilde{\delta}_d$ (i.e. $\{\delta_1, ..., \delta_d\} = \{\widetilde{\delta}_1, ..., \widetilde{\delta}_d\}$) such that these form $\gcd(d, v)$ many *cycles of length* b with respect to h:

$$h: \quad \widetilde{\delta}_{ab+1} \mapsto \widetilde{\delta}_{ab+2} \mapsto \dots \mapsto \widetilde{\delta}_{ab+b} \mapsto \widetilde{\delta}_{ab+1}$$
$$\text{for } a \in \{0, 1, ..., \gcd(d, v) - 1\}.$$

Define

$$\beta_j := \sum_{a=0}^{\gcd(d,v)-1} \widetilde{\delta}_{ab+j} \quad \text{for } j \in \{1, ..., b\}.$$

Then these also form a cycle of length b with respect to h, and their sum is $c\gamma$,

$$h: \ \beta_1 \mapsto \beta_2 \mapsto \dots \mapsto \beta_b \mapsto \beta_1,$$
$$\sum_{j=1}^{b} \beta_j = \sum_{j=1}^{d} \widetilde{\delta}_j = \sum_{j=1}^{d} \delta_j = c\gamma.$$

We obtain

$$\begin{aligned}(H^{(d,c)}, h) \quad &\cong \quad \bigoplus_{a=0}^{\gcd(d,v)-2} \left(\bigoplus_{j=1}^{b} \mathbb{Z} \cdot \widetilde{\delta}_{ab+j}, h \right) \oplus \left(\mathbb{Z} \cdot \gamma + \sum_{j=1}^{b} \mathbb{Z} \cdot \beta_j, h \right) \\ &\cong \quad (\gcd(d, v) - 1)\mathrm{Or}(t^b - 1) \oplus \mathrm{Lo}^{(b,c)}. \qquad \square\end{aligned}$$

Lemma 2.4. *Let $d \in \mathbb{N}$ and $c, \widetilde{c} \in \mathbb{Z}$. The following three conditions are equivalent.*

(i) $\mathrm{Lo}^{(d,c)} \cong \mathrm{Lo}^{(d,\widetilde{c})}$.

(ii) $\mathrm{Lo}^{(d,c)} \oplus \mathrm{Or}(t - 1) \cong \mathrm{Lo}^{(d,\widetilde{c})} \oplus \mathrm{Or}(t - 1)$.

(iii) $\gcd(d, c) = \gcd(d, \widetilde{c})$.

And

$$\begin{aligned} &\mathrm{Lo}^{(d,c)} \oplus \mathrm{Or}(t - 1) \cong \mathrm{Or}(t^d - 1) \oplus \mathrm{Or}(t - 1) \\ \iff \quad &\gcd(d, c) = 1. \end{aligned} \tag{2.3}$$

Proof: Keep the notations $\delta_1, ..., \delta_d, \gamma$ of Definition 2.1 for the elements of $H^{(d,c)}$. And extend them by $\delta_j := \delta_{j_0}$ if $j \in \mathbb{Z} - \{1, ..., d\}$, $j_0 \in \{1, ..., d\}$ and $d|(j - j_0)$.

(iii)⇒(i): We start with $\mathrm{Lo}^{(d,c)}$ for some $c \in \mathbb{N}$ with $c|d$. We will present a construction which leads to certain $\widetilde{c} \in \mathbb{Z}$ with $\mathrm{Lo}^{(d,c)} \cong \mathrm{Lo}^{(d,\widetilde{c})}$. Then we will show that these are all $\widetilde{c} \in \mathbb{Z}$ with $\gcd(d, \widetilde{c}) = c$.

Choose $a \in \mathbb{N}$ with $\gcd(a, d) = 1$, and choose $b \in \mathbb{Z}$. Define

$$\widetilde{\gamma} := \gamma, \quad \widetilde{\delta}_j := b\gamma + \sum_{i=0}^{a-1} \delta_{j+i} \quad \text{for } j \in \mathbb{Z},$$

so that $\widetilde{\delta}_{j_1} = \widetilde{\delta}_{j_2}$ if $d|(j_1 - j_2)$. Of course, $h^{(d,c)}$ acts by

$$h^{(d,c)}: \ \widetilde{\gamma} \mapsto \widetilde{\gamma}, \quad \widetilde{\delta}_j \mapsto \widetilde{\delta}_{j+1}, \tag{2.4}$$

and we have

$$\sum_{j=1}^{d} \widetilde{\delta}_j = (bd + ac) \cdot \widetilde{\gamma}. \tag{2.5}$$

Furthermore, the condition $\gcd(a,d)=1$ implies

$$\mathbb{Z}\cdot\widetilde{\gamma}+\sum_{j=1}^{d}\mathbb{Z}\cdot\widetilde{\delta}_j = H^{(d,c)}\left(=\mathbb{Z}\cdot\gamma+\sum_{j=1}^{d}\mathbb{Z}\cdot\delta_j\right). \tag{2.6}$$

To see this, choose $b_1,b_2\in\mathbb{N}$ with $ab_1-db_2=1$. Then for $j\in\mathbb{Z}$

$$-(b_1b+b_2c)\widetilde{\gamma}+\sum_{k=0}^{b_1-1}\widetilde{\delta}_{j+ak} = -b_1b\widetilde{\gamma}-\sum_{k=1}^{db_2}\delta_{j+k}+b_1b\widetilde{\gamma}+\sum_{k=0}^{db_2}\delta_{j+k}=\delta_j,$$

which shows (2.6). Together, (2.4), (2.5) and (2.6) give

$$\mathrm{Lo}^{(d,c)} \quad\cong\quad \mathrm{Lo}^{(d,bd+ac)}. \tag{2.7}$$

Now choose any $\widetilde{c}\in\mathbb{Z}$ with $\gcd(d,\widetilde{c})=c$. It remains to see that there exist $a\in\mathbb{N}$ and $b\in\mathbb{Z}$ with $\gcd(a,d)=1$ and $\widetilde{c}=bd+ac$.

For any integer $m\in\mathbb{Z}-\{0\}$, write $m=\frac{m}{|m|}\cdot\prod_{p\text{ prime number}}p^{v_p(m)}$, where $v_p(m)\in\mathbb{Z}_{\geq 0}$. We choose

$$\begin{aligned}\widetilde{b} &:= \prod_{p\text{ prime number with }v_p(d)=v_p(\widetilde{c})>0} p,\\ b &:= -\widetilde{b}-|\widetilde{c}|\in\mathbb{Z}_{<0},\\ a &:= \frac{\widetilde{c}}{c}-b\cdot\frac{d}{c}=\frac{\widetilde{c}}{c}+\widetilde{b}\frac{d}{c}+\frac{|\widetilde{c}|}{c}d\in\mathbb{N}.\end{aligned}$$

Then $\widetilde{c}=bd+ac$. For a prime number p with $v_p(d)>v_p(\widetilde{c})$, $v_p(\frac{\widetilde{c}}{c})=0$ and $v_p(\frac{d}{c})>0$ and $v_p(a)=0$. For a prime number p with $v_p(d)=v_p(\widetilde{c})>0$ $v_p(\frac{\widetilde{c}}{c})=0$ and $v_p(b)>0$ and $v_p(a)=0$. For a prime number p with $0<v_p(d)<v_p(\widetilde{c})$, $v_p(\frac{\widetilde{c}}{c})>0$ and $v_p(b\frac{d}{c})=0$ and $v_p(a)=0$. Therefore $\gcd(a,d)=1$.

(i)$\Rightarrow$(ii): This is trivial.

(ii)$\Rightarrow$(iii): We proved already (iii)$\Rightarrow$(i) and (i)$\Rightarrow$(ii). Therefore, in (ii) we can suppose $c=\gcd(d,c)$ and $\widetilde{c}=\gcd(d,\widetilde{c})$. Then we have to show $c=\widetilde{c}$.

Write the elements in Definition 2.1 for $H^{(d,\widetilde{c})}$ with a tilde, so as $\widetilde{\delta}_1,...,\widetilde{\delta}_d,\widetilde{\gamma}$. Write generators of $\mathrm{Or}(t-1)$ on the left-hand side respectively right-hand side of (ii) as β respectively $\widetilde{\beta}$. The automorphisms of the left-hand side respectively right-hand side of (ii) which extend $h^{(d,c)}$ respectively $h^{(d,\widetilde{c})}$ by id on $\mathrm{Or}(t-1)$, are called h respectively $\widetilde{h}$.

Let

$$g:\mathrm{Lo}^{(d,c)}\oplus\mathrm{Or}(t-1)\to\mathrm{Lo}^{(d,\widetilde{c})}\oplus\mathrm{Or}(t-1)$$

be an isomorphism. Then

$$\widetilde{h}\circ g=g\circ h. \tag{2.8}$$

This and $\ker(h-\mathrm{id})=\mathbb{Z}\cdot\gamma\oplus\mathbb{Z}\cdot\beta$ and $\ker(\widetilde{h}-\mathrm{id})=\mathbb{Z}\cdot\widetilde{\gamma}\oplus\mathbb{Z}\cdot\widetilde{\beta}$ imply

$$g(\gamma)=b_1\widetilde{\gamma}+b_2\widetilde{\beta},\quad g(\beta)=b_3\widetilde{\gamma}+b_4\widetilde{\beta}\quad\text{with }\begin{pmatrix}b_1&b_2\\b_3&b_4\end{pmatrix}\in GL(2,\mathbb{Z}).$$

Because of (2.8) and the action of h on $\delta_1, ..., \delta_d$ and $\widetilde{h}$ on $\widetilde{\gamma}$, $\widetilde{\beta}$, $\widetilde{\delta}_1, ..., \widetilde{\delta}_d$, numbers $a_1, ..., a_{d+2} \in \mathbb{Z}$ with

$$g(\delta_j) = a_{d+1}\widetilde{\gamma} + a_{d+2}\widetilde{\beta} + \sum_{i=1}^{d} a_i \widetilde{\delta}_{i+j-1} \quad \text{for } j \in \{1, ..., d\}$$

exist. Write $a := \sum_{i=1}^{d} a_i$. Then

$$\begin{aligned} c(b_1\widetilde{\gamma} + b_2\widetilde{\beta}) &= g(c\gamma) = g(\sum_{j=1}^{d} \delta_j) \\ &= da_{d+1}\widetilde{\gamma} + da_{d+2}\widetilde{\beta} + \left(\sum_{i=1}^{d} a_i\right) \cdot \sum_{j=1}^{d} \widetilde{\delta}_j \\ &= (da_{d+1} + a\widetilde{c})\widetilde{\gamma} + da_{d+2}\widetilde{\beta}, \\ \text{so} \quad cb_1 &= da_{d+1} + a\widetilde{c}, \quad cb_2 = da_{d+2}. \end{aligned}$$

$\pm 1 = b_1b_4 - b_2b_3$ implies $\gcd(b_1, b_2) = 1$. Therefore c is in the ideal generated by d and $\widetilde{c}$, which is the ideal generated by $\widetilde{c}$, because $\widetilde{c}|d$. Thus $\widetilde{c}|c$. Because of the symmetry of the situation also $c|\widetilde{c}$. We obtain $c = \widetilde{c}$. This shows (ii)⇒(iii).

Now the equivalence of (i), (ii) and (iii) is proved. It remains to see (2.3). It follows using the isomorphism $\mathrm{Lo}^{(d,1)} \cong \mathrm{Or}(t^d - 1)$ in Remark 2.2 and the equivalence of (i), (ii) and (iii). □

3. Cells, chains and cycles

Throughout this section we fix a cycle type singularity $f(x_1, ..., x_n)$ as in (1.2) with $n \geq 2$ and $a_1, ..., a_n \in \mathbb{N}$ with (1.3). By [Mi68], the Milnor fiber $f^{-1}(1) \subset \mathbb{C}^n$ and the set

$$\overline{F_0} := f^{-1}(\mathbb{R}_{\geq 0}) \cap S^{2n-1} \subset S^{2n-1} \subset \mathbb{C}^n$$

are diffeomorphic. Cooper [Co82] considers the subset $G \subset \overline{F_0}$ which is defined as follows,

$$G = \{z \in S^{2n-1} \,|\, z_j^{a_j} z_{j+1} \geq 0 \;\forall\; j \in \{1, ..., n-1\},\; z_n^{a_n} z_1 \geq 0\}. \tag{3.1}$$

He conjectures that G is a deformation retract of $\overline{F_0}$. He proves a slightly weaker deformation lemma (stated at the end of 3. in [Co82]) which implies especially that the inclusion map $i_g : G \hookrightarrow \overline{F_0}$ induces epimorphisms in homology. For him and for us, this property suffices.

Cooper builds G up from certain *cells*. We will need these cells, and also refinements of them. For this, quite some notations are needed. They are given now.

Notations 3.1. $N := \{1, ..., n\}$, so that $\mathbb{R}^N = \mathbb{R}^n$. The map

$$\underline{e} : \mathbb{R}^n \to T^n := (S^1)^n \subset \mathbb{C}^n, \quad (r_1, ..., r_n) \mapsto (e^{2\pi i r_1}, ..., e^{2\pi i r_n}),$$

induces an isomorphism $\underline{e}_T : \mathbb{R}^n/\mathbb{Z}^n \to T^n$. Also the projection $\mathrm{pr}_T : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ will be useful. Then $\underline{e} = \underline{e}_T \circ \mathrm{pr}_T$. The following binary operation $\odot$ is not standard, but it will also be useful,

$$\odot : \mathbb{R}^n_{\geq 0} \times (S^1)^n \to \mathbb{C}^n, (a_1, ..., a_n) \odot (b_1, ..., b_n) := (a_1b_1, ..., a_nb_n).$$

Given a finite tuple $(v_1, ..., v_k) \in (\mathbb{R}^n)^k$ of vectors in $\mathbb{R}^n$, we consider the subset of $\mathbb{R}^n$

$$C(v_1, ..., v_k) := \{\sum_{i=1}^{k} t_i v_i \,|\, t_1, ..., t_k \in [0, 1]\}.$$

If $(v_1, ..., v_k)$ is a tuple of linearly independent vectors (which will be the case almost always), $C(v_1, ..., v_k)$ is a hypercube of dimension k. Then it inherits an orientation from the ordered tuple $(v_1, ..., v_k)$. And then its boundary is

$$\begin{aligned}\partial C(v_1, ..., v_k) &= \sum_{i=1}^{k}(-1)^{i-1}C(v_1, ..., \widehat{v_i}, ..., v_k) \\ &\quad -\sum_{i=1}^{k}(-1)^{i-1}(v_i + C(v_1, ..., \widehat{v_i}, ..., v_k)),\end{aligned} \tag{3.2}$$

where $\widehat{v_i}$ means that v_i is erased in the tuple. The following observation will be useful, because we will consider the images of hypercubes under pr_T in $\mathbb{R}^n/\mathbb{Z}^n$. If $v_1 \in \mathbb{Z}^n$, then

$$\begin{aligned}\mathrm{pr}_T(C(v_1, ..., v_k)) &= \mathrm{pr}_T(C(v_1, v_2 + \lambda_2 v_1, ..., v_k + \lambda_k v_1)) \\ &\quad \text{for any } \lambda_2, ..., \lambda_k \in \mathbb{R},\end{aligned} \tag{3.3}$$

and similarly for $v_j \in \mathbb{Z}^n$ instead of $v_1 \in \mathbb{Z}^n$.

The exponents $a_1, ..., a_n \in \mathbb{N}$ in the monomials of the cycle type singularity are the source of useful integers and vectors of integers:

$$\begin{aligned}
\text{Recall} \quad \mu &= a_1 \cdot ... \cdot a_n, \quad d = \mu - (-1)^n. \\
\text{For } k \in N: \quad \underline{a}_k &:= (-1)^{k-1} a_1 \cdot ... \cdot a_{k-1} \in \mathbb{Z}, \quad \text{so } \underline{a}_1 = 1. \\
\underline{a}_{n+1} &:= (-1)^n a_1 \cdot ... \cdot a_n = (-1)^n \mu = (-1)^n d + 1. \\
\text{For } k \in N: \quad \underline{b}_k &:= (\underline{a}_1, ..., \underline{a}_{k-1}, 0, ..., 0) \in \mathbb{Z}^n, \quad \text{so } \underline{b}_1 = (0, ..., 0). \\
\underline{b}_{n+1} &:= (\underline{a}_1, \underline{a}_2, ..., \underline{a}_n) \in \mathbb{Z}^n. \\
\text{For } k \in N: \quad \underline{c}_k &:= (\underline{a}_{n+1}\underline{a}_1, ..., \underline{a}_{n+1}\underline{a}_{k-1}, \underline{a}_k, \underline{a}_{k+1}, ..., \underline{a}_n) \\
&= \underline{a}_{n+1} \cdot \underline{b}_k + (0, ..., 0, \underline{a}_k, ..., \underline{a}_n), \\
&= (-1)^n d \cdot \underline{b}_k + \underline{c}_1 \in \mathbb{Z}^n, \quad \text{especially } \underline{c}_1 = \underline{b}_{n+1}.
\end{aligned}$$

Also the following vectors in $\mathbb{Q}^n$ will be used,

$$\text{for } j \in \{1, ..., d\}: \quad \underline{p}_j := \frac{j}{d} \cdot \underline{c}_1 \in \mathbb{Q}^n.$$

In fact, their images $\mathrm{pr}_T(\underline{p}_j)$ in $\mathbb{R}^n/\mathbb{Z}^n$ will be the only 0-chains in $\mathbb{R}^n/\mathbb{Z}^n$ which we will need. The reason is the equality

$$\mathrm{pr}_T(\underline{p}_{j+1}) = \mathrm{pr}_T(\underline{p}_j + \frac{1}{d}\underline{c}_k) \quad \text{for any } k \in N. \tag{3.4}$$

For any subset $A \subset N$ with $A \neq \emptyset$, define

$$\begin{aligned}
\mathbb{R}^A &:= \{r = (r_1, ..., r_n) \in \mathbb{R}^n \,|\, r_j = 0 \text{ for } j \notin A\} \subset \mathbb{R}^n, \\
\mathbb{Z}^A &\subset \mathbb{Z}^n \text{ and } \mathbb{R}^A_{\geq 0} \subset \mathbb{R}^n_{\geq 0} \text{ and } \mathbb{C}^A \subset \mathbb{C}^n \quad \text{analogously}, \\
\Delta_A &:= \mathbb{R}^A_{\geq 0} \cap S^{2n-1} \subset S^{2n-1}, \\
\mathrm{pr}_A &: \mathbb{R}^n \to \mathbb{R}^A \quad \text{the projection.}
\end{aligned}$$

Δ_A is a deformation of a simplex of dimension $|A| - 1$. We call it *a deformed simplex*. Its boundary consists of the deformed simplices Δ_B with $B \subsetneqq A$.

We want to consider N and its subsets A as cyclic: 1 follows n. In order to write this down, we denote by $(k)_{\mathrm{mod}\, n} \in N$ for $k \in \mathbb{Z}$ the number with $n|(k - (k)_{\mathrm{mod}\, n})$.

For $A \subsetneqq N$ with $A \neq \emptyset$, the *blocks* are the maximal sequences $k, (k+1)_{\text{mod}\, n}, ..., (k+l)_{\text{mod}\, n}$ within A. Their number is called $b(A)$. And the *block beginnings* are the first numbers in the blocks. Explicitly, they are the numbers in the set

$$\{k_1^A, ..., k_{b(A)}^A\} = \{k \in A \,|\, (k-1)_{\text{mod}\, n} \notin A\}, \quad \text{with } 1 \leq k_1^A < ... < k_{b(A)}^A \leq n.$$

The *gaps* of A are the blocks of $N - A$. Also their number is $b(A)$. A set $A \subsetneqq N$ is *thick* if $b(A) = n - |A|$. Equivalent is that each gap of A consists of a single number. For thick $A \subset N$ define the sign

$$\text{sign}(A) := \text{sign}\begin{pmatrix} 1 & ... & |A| & |A|+1 & ... & n \\ \alpha_1 & ... & \alpha_{|A|} & k_1^A & ... & k_{b(A)}^A \end{pmatrix} \in \{\pm 1\} \quad \text{where}$$
$$N - \{k_1^A, ..., k_{b(A)}^A\} = \{\alpha_1, ..., \alpha_{|A|}\} \text{ with } \alpha_1 < ... < \alpha_{|A|}.$$

A set $A \subsetneqq N$ with $A \neq \emptyset$ is *almost thick* if $b(A) = n - |A| - 1$. Equivalent is that one gap consists of two numbers and each other gap consists of a single number. For an almost thick set B with gap $\{k_0, (k_0+1)_{\text{mod}\, n}\}$ of two elements, denote

$$B^{(1)} := B \cup \{k_0\}, \quad B^{(2)} := B \cup \{(k_0+1)_{\text{mod}\, n}\}.$$

$B^{(1)}$ and $B^{(2)}$ are the unique thick sets with $B \subset B^{(i)}$ and $|B^{(i)}| = |B| + 1$. They satisfy $b(B) = b(B^{(i)})$.

For $A \subsetneqq N$ with $A \neq \emptyset$, we will define a subtorus $T_A \subset T^n$ below. For this we define for each block beginning k_j^A in A a vector of integers,

$$\begin{aligned} \underline{d}_j^A &:= \underline{a}_{k_j^A}^{-1}(\underline{b}_{k_{j+1}^A} - \underline{b}_{k_j^A}) \quad \text{for } j \in \{1, ..., b(B)-1\} \\ &= (0, ..., 0, 1, (-a_{k_j^A}), ..., (-a_{k_j^A})...(-a_{k_{j+1}^A - 2}), 0, ..., 0) \in \mathbb{Z}^n, \\ \underline{d}_{b(A)}^A &:= \underline{a}_{k_{b(A)}^A}^{-1}(\underline{a}_{n+1}\underline{b}_{k_1^A} + \underline{c}_1 - \underline{b}_{k_{b(A)}^A}) \\ &= (..., \underline{a}_{n+1}\underline{a}_{k_{b(A)}^A}^{-1}\underline{a}_{k_1^A - 1}, 0, ..., 0, 1, (-a_{k_{b(A)}^A}), ..., \underline{a}_n \underline{a}_{k_{b(A)}^A}^{-1}) \in \mathbb{Z}^n. \end{aligned}$$

Then the subtorus $T_A \subset T^n$ is the set

$$T_A := \underline{e}(C(\underline{d}_1^A, ..., \underline{d}_{b(A)}^A)) \subset T^n.$$

It is a torus of dimension $b(A)$. Finally, observe that the $b(A)$-dimensional hypercube

$$\underline{p}_j + d^{-1}C(\underline{c}_{k_1^A}, ..., \underline{c}_{k_{b(A)}^A})$$

maps by $\underline{e}$ to a subset of T_A,

$$\underline{e}\left(\underline{p}_j + d^{-1}C(\underline{c}_{k_1^A}, ..., \underline{c}_{k_{b(A)}^A})\right) \subset T_A, \tag{3.5}$$

because the vectors $d^{-1}\underline{c}_{k_i^A}$ are linear combinations of the vectors $\underline{d}_1^A, ..., \underline{d}_{b(A)}^A$, and for each vector $\underline{p}_j$ there is a vector $\widetilde{\underline{p}}_j$ such that $\underline{e}(\widetilde{\underline{p}}_j) = \underline{e}(\underline{p}_j)$ and $\widetilde{\underline{p}}_j$ is a linear combination of $\underline{d}_1^A, ..., \underline{d}_{b(A)}^A$. And observe that many of the $2^{b(A)}$ vertices of the image in T_A of this hypercube coincide and that the vertices form the set $\{\underline{e}(\underline{p}_j), \underline{e}(\underline{p}_{(j+1)_{\text{mod}\, d}}), ..., \underline{e}(\underline{p}_{(j+b(A))_{\text{mod}\, d}})\}$, because of (3.4).

Some of these notations are due to Cooper [Co82], namely the deformed simplices Δ_A, the tori T_A, the *block beginnings*, their number $b(A)$ and the *thick* sets A. The hypercubes, the integers $\underline{a}_k$ and the vectors $\underline{b}_k, \underline{c}_k, \underline{d}_k^A$ and $\underline{p}_j$ are new. The observations in the following lemma are all due to Cooper [Co82].

Lemma 3.2. [Co82] *We stick to the cycle type singularity f above and all induced data in the Notations 3.1.*

$$G = \bigcup_{j=1}^{d} \Delta_N \odot \{\underline{e}(\underline{p}_j)\} \cup \bigcup_{A \subsetneqq N,\, A\neq\emptyset} \Delta_A \odot T_A \tag{3.6}$$
$$= \bigcup_{j=1}^{d} \Delta_N \odot \{\underline{e}(\underline{p}_j)\} \cup \bigcup_{A \subsetneqq N \text{ thick}} \Delta_A \odot T_A.$$

For $A \subsetneqq N$ with $A \neq \emptyset$,

$$G \cap \mathbb{C}^A = \Delta_A \odot T_A. \tag{3.7}$$

The sets $\Delta_N \odot \{\underline{e}(\underline{p}_j)\}$ and $\Delta_A \odot T_A$ are called cells *of G. The natural map $\mathrm{Int}(\Delta_A) \times T_A \to \mathrm{Int}(\Delta_A) \odot T_A$ is a diffeomorphism. The natural map $\Delta_N \to \Delta_N \odot \{\underline{e}(\underline{p}_j)\}$ is a diffeomorphism.*

$$\dim_{\mathbb{R}}(\Delta_N \odot \{\underline{e}(\underline{p}_j)\} = n-1, \tag{3.8}$$
$$\dim_{\mathbb{R}}(\Delta_A \odot T_A) = |A| - 1 + b(A) \begin{cases} = n-1 & \text{if } A \text{ is thick}, \\ < n-1 & \text{if } A \text{ is not thick}. \end{cases}$$

If $A, B \subsetneqq N$ with $A \neq B$, then $\mathrm{Int}(\Delta_A) \odot T_A \cap \mathrm{Int}(\Delta_B) \odot T_B = \emptyset$. If $B \subsetneqq A \subsetneqq N$ with $b(B) = b(A)$ then $\Delta_B \odot T_B \subset \Delta_A \odot T_A$.

$A \subset N$ thick implies $|A| \in \{[\frac{n+1}{2}], [\frac{n+1}{2}] + 1, ..., n-1\}$. In the case n even, there are only two thick sets A with $|A| = \frac{n}{2}$, the set $A_{od} := \{1, 3, , ..., n-1\}$ and the set $A_{ev} := \{2, 4, ..., n\}$. The cells $\Delta_{A_{od}} \odot T_{A_{od}}$ and $\Delta_{A_{ev}} \odot T_{A_{ev}}$ are the unit spheres in $\mathbb{C}^{A_{od}}$ respectively $\mathbb{C}^{A_{ev}}$.

$A \subset N$ almost thick implies $|A| \in \{[\frac{n}{2}], ..., n-2\}$.

The proof is easy. We will not give details. If $z_j, z_{j+1} \in \mathbb{C}^*$ with $z_j^{a_j} z_{j+1} > 0$ then

$$\arg(z_{j+1}) \equiv (-a_j)\arg(z_j) \bmod 2\pi.$$

This observation is crucial.

Now we will build up chains, starting with the cells $\Delta_N \odot \{\underline{e}(\underline{p}_j)\}$, and ending with cycles which represent elements of $H_{n-1}(G, \mathbb{Z})$. In section 4 we will show that these cycles (and the cells $\Delta_{A_{od}} \odot T_{A_{od}}$ and $\Delta_{A_{ev}} \odot T_{A_{ev}}$ in the case n even) generate $H_{n-1}(G, \mathbb{Z})$. Of course, it is necessary to refine the cells of G to a simplicial chain complex. We will not describe this precisely. Except from the cells $\Delta_N \odot \{\underline{e}(\underline{p}_j)\}$, we will work only with the following chains (compare (3.5)),

$$\mathcal{C}(A, j) := \Delta_A \odot \underline{e}\left(\underline{p}_j + d^{-1} C(\underline{c}_{k_1^A}, ..., \underline{c}_{k_{b(A)}^A})\right) \subset \Delta_A \odot T_A \tag{3.9}$$
$$\text{for } A \subsetneqq N \text{ thick}, \quad j \in \{1, ..., d\},$$
$$\mathcal{C}(B, A, j) := \Delta_B \odot \underline{e}\left(\underline{p}_j + d^{-1} C(\underline{c}_{k_1^A}, ..., \underline{c}_{k_{b(A)}^A})\right) \subset \Delta_B \odot T_A \tag{3.10}$$
$$\text{for } B \subsetneqq A \subsetneqq N \text{ with } A \text{ thick}, |B| = |A| - 1 \geq 1,\ j \in \{1, ..., d\}.$$

Here our notation is not precise in two ways. (1) If the projection

$$\underline{e} : \underline{p}_j + d^{-1} C(\underline{c}_{k_1^A}, ..., \underline{c}_{k_{b(A)}^A}) \to T^n$$

is not injective, the chain $\mathcal{C}(A, j)$ or $\mathcal{C}(B, A, j)$ shall take multiplicities into account. (2) The chain obtains an orientation from the order of the vectors $\underline{c}_{k_1^A}, ..., \underline{c}_{k_{b(A)}^A}$ if they are linearly independent.

We decompose the boundary of a chain $\mathcal{C}(A,j)$ into two parts, $\partial\mathcal{C}(A,j) = \partial_1\mathcal{C}(A,j)+\partial_2\mathcal{C}(A,j)$ with

$$\partial_1\mathcal{C}(A,j) = \partial\Delta_A \odot \underline{e}\left(\underline{p}_j + d^{-1}C(\underline{c}_{k_1^A},...,\underline{c}_{k_{b(A)}^A})\right), \tag{3.11}$$

$$\begin{aligned}\partial_2\mathcal{C}(A,j) &= (-1)^{|A|-1}\Delta_A \odot \partial\underline{e}\left(\underline{p}_j + d^{-1}C(\underline{c}_{k_1^A},...,\underline{c}_{k_{b(A)}^A})\right) \\ &= (-1)^{|A|-1}\Delta_A \odot \underline{e}\left(\underline{p}_j + d^{-1}\partial C(\underline{c}_{k_1^A},...,\underline{c}_{k_{b(A)}^A})\right).\end{aligned} \tag{3.12}$$

The definitions of the following chains $R_j^{(k)}$ and $X_j^{(k)}$ are crucial. Here $j \in \{1,...,d\}$ and $k \in \{0,1,...,[\frac{n+1}{2}]\}$.

$$R_j^{(k)} := \sum_{A \text{ thick}, |A|=n-k} \operatorname{sign}(A)\cdot\mathcal{C}(A,j) \quad \text{for } k\in\{1,2,...,[\frac{n}{2}]\}, \tag{3.13}$$

$$R_j^{(\frac{n+1}{2})} := 0 \quad \text{if } n \text{ is odd},$$

$$R_j^{(0)} := \Delta_N \odot \{\underline{e}(\underline{p}_j)\}.$$

For $R_j^{(k)}$ with $k\geq 1$, the decomposition $\partial R_j^{(k)} = \partial_1 R_j^{(k)} + \partial_2 R_j^{(k)}$ of its boundary into two parts is well defined. For $k=0$ we define $\partial_2 R_j^{(0)} := 0$ and $\partial_1 R_j^{(0)} := \partial R_j^{(0)}$. The definition of the next chains is inductive. Again $j\in\{1,...,d\}$.

$$X_j^{(0)} := \Delta_N \odot \{\underline{e}(\underline{p}_j)\} = R_j^{(0)}, \tag{3.14}$$

$$X_j^{(k)} := X_j^{(k-1)} - X_{j+1}^{(k-1)} + R_j^{(k)} \quad \text{for } k\in\{1,...,[\frac{n+1}{2}]\}. \tag{3.15}$$

The following theorem is the main result of this section. Its proof takes the rest of this section.

Theorem 3.3. *Again, $j\in\{1,...,d\}$ and $k\in\{0,1,...,[\frac{n+1}{2}]\}$.*

$$\partial X_j^{(k)} = \partial_1 R_j^{(k)}, \tag{3.16}$$

$$\text{and especially} \quad \partial X_j^{([\frac{n+1}{2}])} = 0. \tag{3.17}$$

So, the chains $X_j^{([\frac{n+1}{2}])}$ are cycles and induce homology classes in $H_{n-1}(G,\mathbb{Z})$. (Theorem 4.6 will tell more.) If n is odd then

$$\sum_{j=1}^{d} X_j^{(\frac{n+1}{2})} = 0. \tag{3.18}$$

If n is even then

$$\begin{aligned}\sum_{j=1}^{d} X_j^{(\frac{n}{2})} &= (-1)^{(\frac{n}{2}+2)(\frac{n}{2}+1)\frac{1}{2}}\cdot \underline{a}_1\underline{a}_3...\underline{a}_{n-1} \\ &\cdot \left(\Delta_{A_{od}}\odot T_{A_{od}} + (-1)^{\frac{n}{2}} a_1a_3...a_{n-1}\Delta_{A_{ev}}\odot T_{A_{ev}}\right).\end{aligned} \tag{3.19}$$

Proof: (3.16) for $k=0$ is $\partial X_j^{(0)} = \partial_1 R_j^{(0)}$. It is true by definition of $\partial_1 R_j^{(0)}$. By induction, we obtain for $k\geq 0$

$$\begin{aligned}\partial X_j^{(k+1)} &= \partial X_j^{(k)} - \partial X_{j+1}^{(k)} + \partial R_j^{(k+1)} \\ &= \left(\partial_1 R_j^{(k)} - \partial_1 R_{j+1}^{(k)} + \partial_2 R_j^{(k+1)}\right) + \partial_1 R_j^{(k+1)}.\end{aligned}$$

Therefore we have to show for $k\geq 0$

$$\partial_1 R_j^{(k)} - \partial_1 R_{j+1}^{(k)} + \partial_2 R_j^{(k+1)} = 0. \tag{3.20}$$

In fact, $R_j^{(k+1)}$ was chosen so that (3.20) holds. In order to show (3.20), first we study $\partial_1 R_j^{(k)}$.

For $A=\{k_1,...,k_{|A|}\}\subset N$ with $k_1<...<k_{|A|}$ if $n\notin A$ and with $k_1=n$ & $k_2<...<k_{|A|}$ if $n\in A$ and for $B=\{k_1,...,\widehat{k_j},...,k_{|A|}\}\subset A$ define

$$\mathrm{sign}(B,A) := (-1)^{j-1}. \tag{3.21}$$

Then

$$\begin{aligned}
\partial_1 R_j^{(k)} &= \sum_{A \text{ thick}, |A|=n-k}\ \sum_{B\subset A, |B|=|A|-1} \mathrm{sign}(B,A)\cdot \mathrm{sign}(A)\cdot \mathcal{C}(B,A,j) \\
&\quad \text{for } k\geq 1, \\
\partial_1 R_j^{(0)} &= \sum_{B:\, |B|=n-1} \mathrm{sign}(B,N)\cdot \Delta_B \odot \{\underline{e}(\underline{p}_j)\}.
\end{aligned} \tag{3.22}$$

First we consider the cases $k\geq 1$. A part of the following arguments will be valid also for the case $k=0$ and will give this case. In general, the subsets B of thick sets A with $|B|=|A|-1=n-k-1$ are of three different types,

type I	type II	type III
$b(B)=b(A)+1$	$b(B)=b(A)$	$b(B)=b(A)-1$
$\Rightarrow$ B thick	$\Rightarrow$ B almost thick	

First, consider a set B of type III. Then only one thick set A with $|A|=n-k$ and $A\supset B$ exists. It contains a block $\{k_j^A\}$ which consists of a single number k_j^A, and $B=A-\{k_j^A\}$. The gap of B which contains k_j^A consists of k_j^A and the gaps of A left and right of k_j^A, so this gap of the set B has 3 elements. Therefore the set B is neither thick nor almost thick. All other gaps of B consist of a single number. Observe

$$\begin{aligned}
\mathrm{pr}_B(\underline{c}_{k_j^A}) &= \mathrm{pr}_B(\underline{c}_{k_{j+1}^A}), \quad \text{thus} \\
\mathrm{pr}_B(d^{-1}C(\underline{c}_{k_1^A},...,\underline{c}_{k_{b(A)}^A})) &= \mathrm{pr}_B(d^{-1}C(\underline{c}_{k_1^B},...,\underline{c}_{k_{b(B)}^B})), \quad \text{thus} \\
\mathcal{C}(B,A,j) &= \mathcal{C}(B,j)\subset \Delta_B\odot T_B, \\
\dim \mathcal{C}(B,A,j) &= |B|-1+b(B) = |A|-2+b(A)-1 \\
&= \dim \mathcal{C}(A,j)-2 = \dim R_j^{(k)}-2.
\end{aligned}$$

Therefore this part $\mathcal{C}(B,A,j)$ of the boundary $\partial_1 R_j^{(k)}$ has too small dimension and can be ignored.

Next, consider a set B of type II. It is almost thick. It has one gap which consists of two numbers k_0 and $(k_0+1)_{\mathrm{mod}\, n}$. All other gaps consist of a single number. The only two thick sets A with $A\supset B$ and $|A|=n-k$ are $B^{(1)}:=B\cup\{k_0\}$ and $B^{(2)}:=B\cup\{(k_0+1)_{\mathrm{mod}\, n}\}$. All possibilities for k_0 except one are easy to treat, namely the cases $k_0\in\{1,...,n\}-\{n-1\}$. In all these cases

$$\begin{aligned}
&\mathrm{pr}_B(\underline{c}_{k_i^{B^{(1)}}}) = \mathrm{pr}_B(\underline{c}_{k_i^{B^{(2)}}}) = \mathrm{pr}_B(\underline{c}_{k_i^B}) \text{ for any } i\in\{1,...,b(B)\}, \\
&\qquad \text{thus} \quad \mathcal{C}(B,B^{(1)},j) = \mathcal{C}(B,B^{(2)},j) = \mathcal{C}(B,j).
\end{aligned}$$

One checks also

$$\mathrm{sign}(B,B^{(1)}) = \mathrm{sign}(B,B^{(2)}), \quad \mathrm{sign}(B^{(1)}) = -\mathrm{sign}(B^{(2)}). \tag{3.23}$$

These observations show that these contributions to $\partial_1 R_j^{(k)}$ cancel.

The only difficult possibility for k_0 is the case $k_0 = n-1$. Then

$$\begin{array}{ccccccccc} k_1^{B^{(1)}} = 1 & < & k_2^{B^{(1)}} & < & \ldots & < & k_{b(B)}^{B^{(1)}}, & , & k_j^{B^{(1)}} = k_j^B \\ & & \| & & & & \| & & \\ & & k_1^{B^{(2)}} & < & \ldots & < & k_{b(B)-1}^{B^{(2)}} & < & k_{b(B)}^{B^{(2)}} = n, \end{array}$$

$$\begin{array}{rcl} \underline{c}_{k_{b(B)}^{B^{(2)}}} = \underline{c}_n & = & (-1)^n d \cdot \underline{b}_n + \underline{c}_1, \\ \mathrm{pr}_B(\underline{c}_{k_{b(B)}^{B^{(2)}}}) & = & ((-1)^n d + 1)\,\mathrm{pr}_B(\underline{c}_1), \\ 1 = \mathrm{sign}(B, B^{(2)}) & = & (-1)^{|B|}\mathrm{sign}(B, B^{(1)}), \\ \mathrm{sign}(B^{(2)}) & = & (-1)^{n-1}\mathrm{sign}(B^{(1)}), \\ \mathrm{sign}(B, B^{(2)})\mathrm{sign}(B^{(2)}) & = & (-1)^{b(B)}\mathrm{sign}(B, B^{(1)})\mathrm{sign}(B^{(1)}). \end{array} \tag{3.24}$$

This together with (3.3), formulas for $\underline{c}_k$ and $\underline{d}_j^B$ and especially $k_1^B = 1$ and $\underline{b}_1 = (0, ..., 0)$ shows

$$\begin{array}{cl} & \mathrm{pr}_T\,\mathrm{pr}_B(\underline{p}_j + d^{-1}C(\underline{c}_{k_1^{B^{(2)}}}, ..., \underline{c}_{k_{b(B)}^{B^{(2)}}})) \\ & +(-1)^{b(B)}\,\mathrm{pr}_T\,\mathrm{pr}_B(\underline{p}_j + d^{-1}C(\underline{c}_{k_1^{B^{(1)}}}, ..., \underline{c}_{k_{b(B)}^{B^{(1)}}})) \\ = & \mathrm{pr}_T\,\mathrm{pr}_B(\underline{p}_j + d^{-1}C(\underline{c}_{k_2^B}, ..., \underline{c}_{k_{b(B)}^B}, (-1)^n d\underline{c}_1)) \\ = & \mathrm{pr}_T\, pr_B(\underline{p}_j + C(d^{-1}\underline{c}_{k_2^B}, ..., d^{-1}\underline{c}_{k_{b(B)}^B}, (-1)^n\underline{c}_1)) \\ \overset{(3.3)}{=} & \mathrm{pr}_T\,\mathrm{pr}_B(\underline{p}_j + C(d^{-1}\underline{c}_{k_2^B} - d^{-1}\underline{c}_1, ..., d^{-1}\underline{c}_{k_{b(B)}^B} - d^{-1}\underline{c}_1, (-1)^n\underline{c}_1)) \\ = & \mathrm{pr}_T\,\mathrm{pr}_B(\underline{p}_j + C((-1)^n\underline{b}_{k_2^B}, ..., (-1)^n\underline{b}_{k_{b(B)}^B}, (-1)^n\underline{c}_1)) \\ = & (-1)^{nb(B)}\underline{a}_1\underline{a}_{k_2^B}...\underline{a}_{k_{b(B)}^B} \cdot \mathrm{pr}_T\,\mathrm{pr}_B(C(\underline{d}_1^B, \underline{d}_2^B, ..., \underline{d}_{b(B)}^B)). \end{array}$$

This implies

$$\begin{array}{cl} & \mathcal{C}(B, B^{(2)}, j) + (-1)^{b(B)} \cdot \mathcal{C}(B, B^{(1)}, j) \\ = & (-1)^{nb(B)}\underline{a}_1\underline{a}_{k_2^B}...\underline{a}_{k_{b(B)}^B} \cdot \Delta_B \odot T_B. \end{array} \tag{3.25}$$

This is up to the sign $\mathrm{sign}(B, B^{(2)}) \cdot \mathrm{sign}(B^{(2)})$ the contribution of B to $\partial_1 R_j^{(k)}$. Especially, it is independent of j. Therefore a set B of type II makes the same contribution to $\partial_1 R_j^{(k)}$ and to $\partial_1 R_{j+1}^{(k)}$. Thus its contribution to the difference $\partial_1 R_j^{(k)} - \partial_1 R_{j+1}^{(k)}$ is zero.

Finally, consider a set B of type I. It is thick. Its gaps are the sets

$$\{(k_1^B - 1)_{\mathrm{mod}\, n}\}, ..., \{(k_{b(B)}^B - 1)_{\mathrm{mod}\, n}\}.$$

Exactly $b(B)$ thick sets A with $A \supset B$ and $|A| = n - k$ exist. They are the sets

$$A^{(i)} = B \cup \{(k_i^B - 1)_{\mathrm{mod}\, n}\}$$

for $i \in \{1, ..., b(B)\}$.

The set of block beginnings of $A^{(i)}$ is the set $\{k_1^B, ..., \widehat{k_i^B}, ..., k_{b(B)}^B\}$. Therefore the contribution of B to the boundary $\partial_1 R_j^{(k)}$ is

$$\sum_{i=1}^{b(B)} \quad \mathrm{sign}(B, A^{(i)}) \cdot \mathrm{sign}(A^{(i)}) \\ \cdot \quad \Delta_B \odot \underline{e}\left(\underline{p}_j + d^{-1}C(\underline{c}_{k_1^B}, ..., \widehat{\underline{c}_{k_i^B}}, ..., \underline{c}_{k_{b(B)}^B})\right), \tag{3.26}$$

and the contribution of B to $\partial_1 R_{j+1}^{(k)}$ looks analogously, with j replaced by $j+1$. The following calculation of signs will be useful,

$$\begin{aligned} \mathrm{sign}(B) &= (-1)^{(i-1)+(n-k_i^B)+(b(B)-i)} \cdot \mathrm{sign}(A^{(i)}) \\ &= (-1)^{k_i^B+|B|-1} \cdot \mathrm{sign}(A^{(i)}), \\ \mathrm{sign}(B, A^{(i)}) &= (-1)^{(k_i^B-1)-(i-1)} = (-1)^{k_i^B - i}, \\ \mathrm{sign}(B, A^{(i)}) \cdot \mathrm{sign}(A^{(i)}) &= \mathrm{sign}(B) \cdot (-1)^{|B|+i-1}. \end{aligned} \tag{3.27}$$

On the other side, the boundary $\partial_2 R_j^{(k+1)}$ has only contributions from sets B of type I. The contribution of one such set B is as follows. Here we use (3.12), (3.2) and (3.4).

$$\begin{aligned} & \text{contribution of } B \text{ to } \partial_2 R_j^{(k+1)} \\ \overset{(3.12)}{=} \; & \mathrm{sign}(B)(-1)^{|B|-1} \cdot \Delta_B \odot \underline{e}\left(\underline{p}_j + d^{-1}\partial C(\underline{c}_{k_1^B}, ..., \underline{c}_{k_{b(B)}^B})\right) \\ \overset{(3.2)}{=} \; & \mathrm{sign}(B)(-1)^{|B|-1} \cdot \\ & \Big(\sum_{i=1}^{b(B)} (-1)^{i-1} \Delta_B \odot \underline{e}\left(\underline{p}_j + d^{-1}C(\underline{c}_{k_1^B}, ..., \widehat{\underline{c}_{k_i^B}}, ..., \underline{c}_{k_{b(B)}^B})\right) \\ & - \sum_{i=1}^{b(B)} (-1)^{i-1} \Delta_B \odot \underline{e}\left(\underline{p}_j + d^{-1}\underline{c}_{k_i^B} + d^{-1}C(\underline{c}_{k_1^B}, ..., \widehat{\underline{c}_{k_i^B}}, ..., \underline{c}_{k_{b(B)}^B})\right)\Big) \\ \overset{(3.4)}{=} \; & \sum_{i=1}^{b(B)} \mathrm{sign}(B)(-1)^{|B|+i} \Delta_B \odot \underline{e}\left(\underline{p}_j + d^{-1}C(\underline{c}_{k_1^B}, ..., \widehat{\underline{c}_{k_i^B}}, ..., \underline{c}_{k_{b(B)}^B})\right) \\ & - \sum_{i=1}^{b(B)} \mathrm{sign}(B)(-1)^{|B|+i} \Delta_B \odot \underline{e}\left(\underline{p}_{j+1} + d^{-1}C(\underline{c}_{k_1^B}, ..., \widehat{\underline{c}_{k_i^B}}, ..., \underline{c}_{k_{b(B)}^B})\right) \\ = \; & -(\text{contribution of } B \text{ to } \partial_1 R_j^{(k)}) \\ & +(\text{contribution of } B \text{ to } \partial_1 R_{j+1}^{(k)}). \end{aligned}$$

Therefore the contribution of B to $\partial_1 R_j^{(k)} - \partial_1 R_{j+1}^{(k)} + \partial_2 R_j^{(k+1)}$ is zero.

So, no set $B \subset N$ with $|B| = n-k-1$ gives a contribution to this sum. Therefore this sum is zero. (3.20) is proved for $k \geq 1$.

Now we will show (3.20) for $k = 0$. $\partial_1 R_j^{(0)}$, $\partial_1 R_{j+1}^{(0)}$ and $\partial_2 R_j^{(1)}$ are sums over sets $B \subset N$ with $|B| = n-1$. These sets are the sets $B^{(i)} := N - \{i\}$ for $i \in N$. The set $B^{(i)}$ is thick with $b(B^{(i)}) = 1$. It can be treated as the sets of type I above. The same calculations as above for sets of type I show that the contribution of $B^{(i)}$ to the sum $\partial_1 R_j^{(0)} - \partial_1 R_{j+1}^{(0)} + \partial_2 R_j^{(1)}$ is zero. Therefore this sum is zero. Therefore (3.20) holds also for $k = 0$, so it holds for all $k \geq 0$.

This implies (3.16) for $k\geq 1$. It holds for $k=0$ anyway.

It remains to prove (3.17), (3.18) and (3.19). (3.17) is because of (3.16) equivalent to $\partial_1 R_j^{([\frac{n+1}{2}])}=0$. In the case n odd this is trivial as then $R_j^{(\frac{n+1}{2})}=0$ by definition. In the case n even, we have to show $\partial_1 R_j^{(\frac{n}{2})}=0$. By Lemma 3.2, there are only two thick sets A with $|A|=\frac{n}{2}$, the sets $A_{od}=\{1,3,...,n-1\}$ and $A_{ev}=\{2,4,...,n\}$. In the discussion above of $\partial_1 R_j^{(k)}$ for $k\geq 1$, we distinguished 3 different types of sets B with $|B|=|A|-1=n-k-1$ and $B\subset A$. In the cases $A\in\{A_{od},A_{ev}\}$, we have only sets B of type III. Above we stated that the part $\mathcal{C}(B,A,j)$ of $\partial_1 R_j^{(k)}$ (here for $k=\frac{n}{2}$) has too small dimension and can be ignored. Therefore $\partial_1 R_j^{(\frac{n}{2})}=0$. We proved (3.17).

(3.18) holds for odd n and is an immediate consequence of (3.15) and $R_j^{(\frac{n+1}{2})}=0$.

(3.19) holds for even n. It requires two calculations which are similar to the one which led to (3.25). The details are as follows. (3.15) and (3.13) show

$$\begin{aligned}
\sum_{j=1}^{d} X_j^{(\frac{n}{2})} &= \sum_{j=1}^{d} R_j^{(\frac{n}{2})} \\
&= \sum_{j=1}^{d}\mathrm{sign}(A_{od})\cdot\mathcal{C}(A_{od},j)+\sum_{j=1}^{d}\mathrm{sign}(A_{ev})\cdot\mathcal{C}(A_{ev},j) \\
&= \sum_{j=1}^{d}\mathrm{sign}(A_{od})\cdot\Delta_{A_{od}}\odot\underline{e}\left(\underline{p}_j+d^{-1}C(\underline{c}_1,\underline{c}_3,...,\underline{c}_{n-1})\right) \\
&+ \sum_{j=1}^{d}\mathrm{sign}(A_{ev})\cdot\Delta_{A_{ev}}\odot\underline{e}\left(\underline{p}_j+d^{-1}C(\underline{c}_2,\underline{c}_4,...,\underline{c}_{n})\right).
\end{aligned}$$

Here

$$\begin{aligned}
\mathrm{sign}(A_{od}) &= (-1)^{\frac{n}{2}(\frac{n}{2}+1)\frac{1}{2}}, \\
\mathrm{sign}(A_{ev}) &= (-1)^{(\frac{n}{2}-1)\frac{n}{2}\frac{1}{2}}=(-1)^{\frac{n}{2}}\cdot\mathrm{sign}(A_{od}).
\end{aligned}$$

Because of $\underline{p}_{j+1}=\underline{p}_j+d^{-1}\underline{c}_1$ and (3.3) (and $(-1)^n=1$ as n is even), we have

$$\begin{aligned}
&\sum_{j=1}^{d}\mathrm{pr}_T\left(\underline{p}_j+d^{-1}C(\underline{c}_1,\underline{c}_3,...,\underline{c}_{n-1})\right) \\
=\ &\mathrm{pr}_T\left(\underline{p}_1+C(\underline{c}_1,d^{-1}\underline{c}_3,....,d^{-1}\underline{c}_{n-1})\right) \\
\overset{(3.3)}{=}\ &\mathrm{pr}_T\left(\underline{p}_1+C(\underline{c}_1,d^{-1}\underline{c}_3-d^{-1}\underline{c}_1,...,d^{-1}\underline{c}_{n-1}-d^{-1}\underline{c}_1)\right) \\
=\ &\mathrm{pr}_T\left(\underline{p}_1+C(\underline{c}_1,(-1)^n\underline{b}_3,...,(-1)^n\underline{b}_{n-1})\right) \\
=\ &(-1)^{n/2-1}\,\mathrm{pr}_T\left(C(\underline{b}_3,...,\underline{b}_{n-1},\underline{c}_1)\right).
\end{aligned}$$

If we identify $\mathrm{pr}_{A_{od}}(\underline{b}_j)$ with a column vector in $M_{n/2\times 1}(\mathbb{Z})$ then the tuple $(\mathrm{pr}_{A_{od}}(\underline{b}_3),...,\mathrm{pr}_{A_{od}}(\underline{b}_{n-1}),\mathrm{pr}_{A_{od}}(\underline{c}_1))$ is an upper triangular $\frac{n}{2}\times\frac{n}{2}$-matrix, and its determinant is $\underline{a}_1\underline{a}_3...\underline{a}_{n-1}$. Therefore

$$\begin{aligned} & \sum_{j=1}^{d}\Delta_{A_{od}}\odot\underline{e}\left(\underline{p}_j+d^{-1}C(\underline{c}_1,\underline{c}_3,...,\underline{c}_{n-1})\right) \\ = \ & (-1)^{n/2-1}\underline{a}_1\underline{a}_3...\underline{a}_{n-1}\cdot\Delta_{A_{od}}\odot T_{A_{od}}. \end{aligned}$$

Observe

$$\mathrm{pr}_{A_{ev}}(\underline{c}_{2l})=\mathrm{pr}_{A_{ev}}(\underline{c}_{2l-1}).$$

Therefore

$$\begin{aligned} & \sum_{j=1}^{d}\mathrm{pr}_{A_{ev}}\,\mathrm{pr}_T\left(\underline{p}_j+d^{-1}C(\underline{c}_2,\underline{c}_4,...,\underline{c}_n)\right) \\ = \ & \sum_{j=1}^{d}\mathrm{pr}_{A_{ev}}\,\mathrm{pr}_T\left(\underline{p}_j+d^{-1}C(\underline{c}_1,\underline{c}_3,...,\underline{c}_{n-1})\right) \\ = \ & (-1)^{n/2-1}\,\mathrm{pr}_{A_{ev}}\,\mathrm{pr}_T\left(C(\underline{b}_3,...,\underline{b}_{n-1},\underline{c}_1)\right). \end{aligned}$$

If we identify $\mathrm{pr}_{A_{ev}}(\underline{b}_j)$ with a column vector in $M_{n/2\times 1}(\mathbb{Z})$ then the tuple $(\mathrm{pr}_{A_{ev}}(\underline{b}_3),...,\mathrm{pr}_{A_{ev}}(\underline{b}_{n-1}),\mathrm{pr}_{A_{ev}}(\underline{c}_1))$ is an upper triangular $\frac{n}{2}\times\frac{n}{2}$-matrix, and its determinant is

$$\underline{a}_2\underline{a}_4...\underline{a}_n=\underline{a}_1\underline{a}_3...\underline{a}_{n-1}\cdot a_1a_3...a_{n-1}.$$

Therefore

$$\begin{aligned} & \sum_{j=1}^{d}\Delta_{A_{ev}}\odot\underline{e}\left(\underline{p}_j+d^{-1}C(\underline{c}_2,\underline{c}_4,...,\underline{c}_n)\right) \\ = \ & (-1)^{n/2-1}\underline{a}_1\underline{a}_3...\underline{a}_{n-1}\cdot a_1a_3...a_{n-1}\cdot\Delta_{A_{ev}}\odot T_{A_{ev}}. \end{aligned}$$

(3.19) follows. □

4. A spectral sequence, following and correcting Cooper

Throughout this section we fix a cycle type singularity $f(x_1,...,x_n)$ as in (1.2) with $n\geq 2$ and $a_1,...,a_n\in\mathbb{N}$ with (1.3), as in section 3. Cooper [Co82] considered a filtration of the set $G\subset\overline{F_0}=f^{-1}(\mathbb{R}_{\geq 0})\cap S^{2n-1}\subset S^{2n-1}\subset\mathbb{C}^n$, which was defined in (3.1), by a sequence of subsets,

$$G=G_n\supset G_{n-1}\supset ...\supset G_1\supset G_0=\emptyset.$$

He studied a spectral sequence $(E^r_{s,t},d^r_{s,t})_{r\geq 1,s\geq 1,s+t\geq 0}$ which is associated to this filtration. It allows to determine $H_{n-1}(G,\mathbb{Z})$. Though he made two serious mistakes. Here we will partly follow his line of thoughts, but correct the mistakes. Remark 5.1 will explain the differences to [Co82].

Cooper does not give a reference for the construction of the spectral sequence. We found a manuscript of Hatcher [Ha04] very useful for our situation and will cite Theorem 4.2 and some preparations for Theorem 4.2 from it.

First we will define the subspaces $G_s\subset G$ and make some elementary observations. Then we will introduce the spectral sequence, following Hatcher. Then we will study it in detail (following partly Cooper) in Lemma 4.3 and Theorem 4.5. Theorem 4.6 will give the conclusion for $H_{n-1}(G,\mathbb{Z})$. It complements Theorem 3.3.

Definition 4.1. [Co82] Recall the definition (3.1) of the set $G \subset \mathbb{C}^n$. Define $G_n := G$, $G_0 := \emptyset$, and define for $s \in \{1, ..., n-1\}$

$$\begin{aligned} G_s &:= \{z \in G \,|\, \text{at most } s \text{ of the coordinates } z_1, ..., z_n \text{ are } \neq 0\} \\ &= \bigcup_{A \subset N:\, |A| \leq s} \Delta_A \odot T_A. \end{aligned} \tag{4.1}$$

Here the second equality follows from Lemma 3.2. Lemma 3.2 gives also the differences $G_s - G_{s-1}$,

$$G_n - G_{n-1} = \bigcup_{j=1}^{d} \mathrm{Int}(\Delta_N) \odot \{\underline{e}(\underline{p}_j)\}, \tag{4.2}$$

$$G_s - G_{s-1} = \bigcup_{A \subset N:\, |A| = s} \mathrm{Int}(\Delta_A) \odot T_A \text{ for } s \in \{1, ..., n-1\}. \tag{4.3}$$

In this paper, all considered homology groups will have coefficients in $\mathbb{Z}$. Especially, we consider for $(s,t) \in N \times \mathbb{Z}$ the homology groups of the spaces G_s and of the pairs (G_s, G_{s-1}),

$$A^1_{s,t} := H_{s+t}(G_s) := H_{s+t}(G_s, \mathbb{Z}), \tag{4.4}$$

$$E^1_{s,t} := H_{s+t}(G_s, G_{s-1}) := H_{s+t}(G_s, G_{s-1}, \mathbb{Z}). \tag{4.5}$$

We extend this definition by

$$\begin{aligned} A^1_{s,t} = E^1_{s,t} = 0 \qquad &\text{for } s \leq 0, \\ A^1_{s,t} = H_{s+t}(G),\ E^1_{s,t} = 0 \qquad &\text{for } s \geq n+1. \end{aligned}$$

Of course

$$A^1_{s,t} = E^1_{s,t} = 0 \qquad \text{for } s+t < 0 \text{ or } s+t \geq n, \tag{4.6}$$

as $\dim_{\mathbb{R}} G_s \leq n-1$. Cooper [Co82] observed that Lemma 3.2 (especially that the map

$$\mathrm{Int}(\Delta_A) \times T_A \to \mathrm{Int}(\Delta_A) \odot T_A$$

is a diffeomorphism) and (4.3) imply

$$H_*(G_n, G_{n-1}) = \bigoplus_{j=1}^{d} H_{n-1}(\Delta_N \odot \{\underline{e}(\underline{p}_j)\}, \partial\Delta_N \odot \{\underline{e}(\underline{p}_j)\}), \tag{4.7}$$

$$H_*(G_s, G_{s-1}) = \bigoplus_{A \subset N:\, |A| = s} H_*(\Delta_A \odot T_A, \partial\Delta_A \odot T_A) \quad \text{for } s \leq n-1. \tag{4.8}$$

And here

$$H_{n-1}(\Delta_N \odot \{\underline{e}(\underline{p}_j)\}, \partial\Delta_N \odot \{\underline{e}(\underline{p}_j)\}) \cong \mathbb{Z} \text{ with generator the class } [X_j^{(0)}], \tag{4.9}$$

$$H_*(\Delta_A \odot T_A, \partial\Delta_A \odot T_A) \cong H_{|A|-1}(\Delta_A, \partial\Delta_A) \otimes H_*(T_A), \tag{4.10}$$

$$H_{|A|-1}(\Delta_A, \partial\Delta_A) \cong \mathbb{Z} \text{ with generator the class } [\Delta_A], \tag{4.11}$$

$$H_*(T_A) \cong \bigotimes_{j=1}^{b(A)} H_*(S^1), \quad H_*(S^1) = H_0(S^1) \oplus H_1(S^1), \tag{4.12}$$

$$H_0(S^1) \cong \mathbb{Z} \text{ and } H_1(S^1) \cong \mathbb{Z} \text{ with generators } [p] \text{ and } [S^1] \text{ for any } p \in S^1. \tag{4.13}$$

By (4.7) and (4.9) the group

$$E^1_{n,-1} = H_{n-1}(G_n, G_{n-1}) = H_*(G_n, G_{n-1}) = \bigoplus_{j=1}^{d} \mathbb{Z} \cdot [X_j^{(0)}] \tag{4.14}$$

is a $\mathbb{Z}$-lattice of rank d with generators the classes $[X_1^{(0)}], ..., [X_d^{(0)}]$. All the groups $E^1_{s,t}$ are $\mathbb{Z}$-lattices (= finitely generated free $\mathbb{Z}$-modules).

The long exact homology sequence for the pair (G_s, G_{s-1}) for $s \in N$ reads as follows,

$$\cdots A^1_{s-1,t+1} \xrightarrow{i^1} A^1_{s,t} \xrightarrow{j^1} E^1_{s,t} \xrightarrow{k^1} A^1_{s-1,t} \xrightarrow{i^1} A^1_{s,t-1} \xrightarrow{j^1} E^1_{s,t-1} \xrightarrow{k^1} A^1_{s-1,t-1} \cdots$$

Here i^1 is induced by the embedding $G_{s-1} \hookrightarrow G_s$, j^1 is the natural map from absolute homology to relative homology, and k^1 is the boundary map. Together, these exact homology sequences can be put into a large diagram which Hatcher [Ha04] calls a *staircase diagram*, because the long exact sequences look like staircases in this diagram:

$$\begin{array}{ccccccccc}
A^1_{s-1,t+1} & \xrightarrow{j^1} & E^1_{s-1,t+1} & \xrightarrow{k^1} & A^1_{s-2,t+1} & \xrightarrow{j^1} & E^1_{s-2,t+1} & \xrightarrow{k^1} & A^1_{s-3,t+1} \\
\downarrow i^1 & & & & \downarrow i^1 & & & & \downarrow i^1 \\
A^1_{s,t} & \xrightarrow{j^1} & E^1_{s,t} & \xrightarrow{k^1} & A^1_{s-1,t} & \xrightarrow{j^1} & E^1_{s-1,t} & \xrightarrow{k^1} & A^1_{s-2,t} \\
\downarrow i^1 & & & & \downarrow i^1 & & & & \downarrow i^1 \\
A^1_{s+1,t-1} & \xrightarrow{j^1} & E^1_{s+1,t-1} & \xrightarrow{k^1} & A^1_{s,t-1} & \xrightarrow{j^1} & E^1_{s,t-1} & \xrightarrow{k^1} & A^1_{s-1,t-1}
\end{array}$$

The map

$$d^1_{s,t} := d^1 := j^1 \circ k^1 : E^1_{s,t} \to E^1_{s-1,t} \tag{4.15}$$

is the cellular boundary map [Ha04]. The system $(E^1_{s,t}, d^1_{s,t})_{s,t\in\mathbb{Z}}$ of spaces and maps is the first page of a spectral sequence which is associated to the filtration

$$G = G_n \supset G_{n-1} \supset ... \supset G_1 \supset G_0 = \emptyset.$$

The other pages $(E^r_{s,t}, d^r_{s,t})_{s,t\in\mathbb{Z}}$ for $r \geq 2$ are constructed inductively together with versions of the staircase diagram above for any $r \geq 2$ instead of $r = 1$, so, tuples $(A^r_{s,t}, E^r_{s,t}, i^r, j^r, k^r, d^r)$ are constructed inductively for any $r \geq 2$. The following theorem describes this construction and gives the general properties. It follows from Lemma 5.1 and Proposition 5.2 in [Ha04]. Proposition 5.2 in [Ha04] applies with the conditions (i) and (ii) in it. The indices in $A^1_{s,t}$ and $E^1_{s,t}$ here are chosen differently (more standard) from those in [Ha04].

Theorem 4.2. [Ha04, Lemma 5.1 and Proposition 5.2]

(a) (Properties) Fix $r \geq 2$. The tuple $(A^r_{s,t}, E^r_{s,t}, i^r, j^r, k^r, d^r)$ will be constructed from the tuple $(A^{r-1}_{s,t}, E^{r-1}_{s,t}, i^{r-1}, j^{r-1}, k^{r-1}, d^{r-1})$ in part (c). It has the following properties:

$$\begin{aligned} i^r : A^r_{s,t} &\to A^r_{s+1,t-1}, && (4.16)\\ j^r : A^r_{s,t} &\to E^r_{s-r+1,t+r-1}, && (4.17)\\ k^r : E^r_{s,t} &\to A^r_{s-1,t}, && (4.18)\\ d^r := j^r \circ k^r : E^r_{s,t} &\to E^r_{s-r,t+r-1}, \quad d^r \circ d^r = 0. && (4.19)\end{aligned}$$

The restriction of i^r to $A^r_{s,t}$ is also called $i^r_{s,t}$, and similarly for j^r, k^r and d^r. There are long exact sequences,

$$\begin{aligned} \cdots A^r_{s+r-2,t-r+2} \xrightarrow{i^r} A^r_{s+r-1,t-r+1} \xrightarrow{j^r} E^r_{s,t} \xrightarrow{k^r} A^r_{s-1,t} \\ \xrightarrow{i^r} A^r_{s,t-1} \xrightarrow{j^r} E^r_{s-r+1,t+r-2} \xrightarrow{k^r} A^r_{s-r,t+r-2} \cdots \end{aligned}$$

They can be put together into the following staircase diagram (the positions of the arrows for j^r are not precise).

$$\begin{array}{ccccccccc} A^r_{s-1,t+1} & \overset{\nearrow}{j^r} & E^r_{s-1,t+1} & \xrightarrow{k^r} & A^r_{s-2,t+1} & \overset{\nearrow}{j^r} & E^r_{s-2,t+1} & \xrightarrow{k^r} & A^r_{s-3,t+1} \\ \downarrow i^r & & & & \downarrow i^r & & & & \downarrow i^r \\ A^r_{s,t} & \overset{\nearrow}{j^r} & E^r_{s,t} & \xrightarrow{k^r} & A^r_{s-1,t} & \overset{\nearrow}{j^r} & E^r_{s-1,t} & \xrightarrow{k^r} & A^r_{s-2,t} \\ \downarrow i^r & & & & \downarrow i^r & & & & \downarrow i^r \\ A^r_{s+1,t-1} & \overset{\nearrow}{j^r} & E^r_{s+1,t-1} & \xrightarrow{k^r} & A^r_{s,t-1} & \overset{\nearrow}{j^r} & E^r_{s,t-1} & \xrightarrow{k^r} & A^r_{s-1,t-1} \end{array}$$

(b) (Spectral sequence) The part $(E^r_{s,t}, d^r_{s,t})_{r\geq 1, s,t\in\mathbb{Z}}$ of the tuples above for all $r \geq 1$ is a spectral sequence. It converges to $H_*(G)$. This imprecise statement means the following. For each $(s,t) \in \mathbb{Z}^2$, a bound $r(s,t) \in \mathbb{N}$ exists such that the space $E^\infty_{s,t} := E^{r(s,t)}_{s,t}$ coincides with $E^r_{s,t}$ for any $r \geq r(s,t)$, and

$$\begin{aligned} E^\infty_{s,t} &\cong F^s_{s+t}/F^{s-1}_{s+t}, \text{ where} && (4.20)\\ F^s_{s+t} &:= \operatorname{Im}(A^1_{s,t} \to A^1_{n,s+t-n}) = \operatorname{Im}(H_{s+t}(G_s) \to H_{s+t}(G)). \end{aligned}$$

(c) (Construction) Fix $r \geq 2$. Suppose that the tuple $(A^{r-1}_{s,t}, E^{r-1}_{s,t}, i^{r-1}, j^{r-1}, k^{r-1}, d^{r-1})$ has been constructed. Then especially $d^{r-1} \circ d^{r-1} = 0$. Therefore the quotient

$$E^r_{s,t} := \ker(d^{r-1}_{s,t}) / \operatorname{Im}(d^{r-1}_{s+r-1,t-r+2}) \tag{4.21}$$

is well-defined. It is a subquotient of $E^{r-1}_{s,t}$. The space

$$A^r_{s,t} := i^{r-1}(A^{r-1}_{s-1,t+1}) \subset A^{r-1}_{s,t} \tag{4.22}$$

is well-defined, anyway. It is a subspace of $A^{r-1}_{s,t}$. The map i^r is the restriction of i^{r-1} to $A^r_{s,t}$. The map j^r on $A^r_{s,t} = i^{r-1}(A^{r-1}_{s-1,t+1})$ is defined on $i^{r-1}(a)$ for $a \in A^{r-1}_{s-1,t+1}$ by

$$j^r(i^{r-1}(a)) := [j^{r-1}(a)].$$

A priori, this is in $E^{r-1}_{s-r+1,t+r-1}/\operatorname{Im}(d^{r-1}_{s,t+1})$. But it turns out to be in $E^r_{s-r+1,t+r-1}$. The map k^r on $E^r_{s,t}$ is defined on $[a]$ for $a \in \ker(d^{r-1}_{s,t}) \subset E^{r-1}_{s,t}$ by $k^r([a]) := k^{r-1}(a)$ which is a priori in $A^{r-1}_{s-1,t}$. But it turns out to be in $A^r_{s-1,t}$ and to be well-defined.

Theorem 4.2 gives basic and general properties, which are not specific to the geometry of G and its subspaces G_s (except for our definition of F^s_{s+t} where we used $A^1_{n,s+t-n} = H_{s+t}(G)$). Now we will use the geometry of G and its subspaces G_s to make more specific statements. We are interested especially in $E^\infty_{s,n-1-s}$ as that is the subquotient F^s_{n-1}/F^{s-1}_{n-1} of $H_{n-1}(G)$ by part (b). Lemma 4.3 gives first elementary observations. Algebraic statements in Lemma 4.4 make a part of Theorem 4.5 more transparent. Theorem 4.5 states the main results on the spectral sequence. Theorem 4.6 gives the conclusion for $H_{n-1}(G)$. It complements Theorem 3.3.

Lemma 4.3. *(a) For $r \geq 2$*

$$E^r_{s,n-1-s} = \ker(d^{r-1}_{s,n-1-s}) \subset E^{r-1}_{s,n-1-s} \subset ... \subset E^1_{s,n-1-s}. \tag{4.23}$$

(b) For $r \geq 1$ and for $s < [\frac{n+1}{2}]$

$$E^r_{s,n-1-s} = E^\infty_{s,n-1-s} = 0 \quad \text{and thus } F^s_{n-1} = 0. \tag{4.24}$$

(c) Consider even n. For $r \geq 1$

$$\begin{aligned} F^{n/2}_{n-1} &= \operatorname{Im}\big((H_{n-1}(G_{\frac{n}{2}}) \to H_{n-1}(G)\big) \cong E^\infty_{\frac{n}{2},\frac{n}{2}-1} \\ &= E^r_{\frac{n}{2},\frac{n}{2}-1} = E^1_{\frac{n}{2},\frac{n}{2}-1} = H_{n-1}(G_{\frac{n}{2}}, G_{\frac{n}{2}-1}) \cong \mathbb{Z}^2. \end{aligned} \tag{4.25}$$

The generators are the classes in $H_{n-1}(G)$ respectively in $H_{n-1}(G_{\frac{n}{2}}, G_{\frac{n}{2}-1})$ of the spheres $\Delta_{A_{od}} \odot T_{A_{od}} \subset \mathbb{C}^{A_{od}}$ and $\Delta_{A_{ev}} \odot T_{A_{ev}} \subset \mathbb{C}^{A_{ev}}$ (see Lemma 3.2).

Proof: (a) Because of (4.6) $E^{r-1}_{s,t} = 0$ for $s+t \geq n$. Therefore $d^{r-1}_{s+r-1,n-1-s-r+2} = 0$, and by (4.21) $E^r_{s,n-1-s} = \ker(d^{r-1}_{s,n-1-s})$.

(b) For $s \leq n-1$, the set $E^1_{s,n-1-s} = H_{n-1}(G_s, G_{s-1})$ has by (4.8), (4.10) and (4.12) only contributions from sets $A \subset N$ with $b(A) = n - |A|$, so from thick sets A. By Lemma 3.2, if $s < [\frac{n+1}{2}]$, there are no thick sets A with $|A| = s$. Therefore then $E^1_{s,n-1-s} = 0$ and $E^r_{s,n-1-s} = 0$ for $r \geq 1$ and $E^\infty_{s,n-1-s} = 0$. This and (4.20) imply inductively $F^s_{n-1} = 0$ for $s < [\frac{n+1}{2}]$.

(c) The first equality is the definition of $F^{n/2}_{n-1}$. The isomorphism $F^{n/2}_{n-1} \cong E^\infty_{\frac{n}{2},\frac{n}{2}-1}$ follows with (4.20) and $F^s_{n-1} = 0$ for $s < \frac{n}{2}$.

The equality $E^1_{\frac{n}{2},\frac{n}{2}-1} = H_{n-1}(G_{\frac{n}{2}}, G_{\frac{n}{2}-1})$ is the definition of $E^1_{\frac{n}{2},\frac{n}{2}-1}$. By (4.8), (4.10) and (4.12) it has a contribution isomorphic to $\mathbb{Z}$ with generator $[\Delta_A \odot T_A]$ for any thick set A with $|A| = \frac{n}{2}$. By Lemma 3.2, these are only the sets A_{od} and A_{ev}. Therefore $E^1_{\frac{n}{2},\frac{n}{2}-1} \cong \mathbb{Z}^2$. As the spheres $\Delta_{A_{od}} \odot T_{A_{od}}$ and $\Delta_{A_{ev}} \odot T_{A_{ev}}$ have no boundary, the boundary map

$$k^1_{\frac{n}{2},\frac{n}{2}-1} : E^1_{\frac{n}{2},\frac{n}{2}-1} \to A^1_{\frac{n}{2}-1,\frac{n}{2}-1} = H_{n-2}(G_{\frac{n}{2}-1})$$

and also the induced maps $k^r_{\frac{n}{2},\frac{n}{2}-1}$ and the maps $d^r_{\frac{n}{2},\frac{n}{2}-1}$ are zero maps. Therefore for $r \geq 1$

$$E^1_{\frac{n}{2},\frac{n}{2}-1} = E^r_{\frac{n}{2},\frac{n}{2}-1} = E^\infty_{\frac{n}{2},\frac{n}{2}-1}.$$

Part (c) is proved. □

The sole purpose of the following elementary algebraic lemma is to make Theorem 4.5 (d) more transparent.

Lemma 4.4. *Recall from (4.14) that the group $E^1_{n,-1} = H_{n-1}(G, G_{n-1})$ is a $\mathbb{Z}$-lattice of rank d with generators the classes $[X^{(0)}_1], ..., [X^{(0)}_d]$. Define for $r \in \{1, ..., [\frac{n+3}{2}]\}$ the sublattice*

$$\widetilde{E}^r_{n,-1} := \sum_{j=1}^d \mathbb{Z} \cdot [X^{(r-1)}_j] \subset E^1_{n,-1}, \tag{4.26}$$

which is generated by the classes $[X_1^{(r-1)}], ..., [X_d^{(r-1)}]$ *in* $H_{n-1}(G, G_{n-1})$. *Then* $\widetilde{E}^1_{n,-1} = E^1_{n,-1}$. *For* $r \geq 2$

$$[X_j^{(r-1)}] = [X_j^{(r-2)}] - [X^{(r-2)}_{(j+1)_{\mathrm{mod}\, d}}] \quad \text{for } j \in \{1, ..., d\}, \tag{4.27}$$

$$\sum_{j=1}^{d} [X_j^{(r-1)}] = 0, \tag{4.28}$$

and $\widetilde{E}^r_{n,-1} \subset E^1_{n,-1}$ *is a sublattice of rank* $d-1$ *which is generated by any* $d-1$ *of the* d *elements* $[X_1^{(r-1)}], ..., [X_d^{(r-1)}]$. *The lattice* $\widetilde{E}^2_{n,-1}$ *is a primitive sublattice of* $E^1_{n,-1}$. *For* $r \geq 3$, $\widetilde{E}^r_{n,-1}$ *has index* d *in* $\widetilde{E}^{r-1}_{n,-1}$.

Proof: (4.27) follows from the definition of $X_j^{(r-1)}$ in (3.15) and from $R_j^{(r-1)} \subset G_{n-1}$. (4.28) is an immediate consequence of (4.27). $\widetilde{E}^2_{n,-1}$ is obviously a primitive sublattice of $E^1_{n,-1}$ of rank $d-1$. For $r \geq 3$, $\widetilde{E}^r_{n,-1}$ has index d in $\widetilde{E}^{r-1}_{n,-1}$ because

$$\widetilde{E}^r_{n,-1} = \{\sum_{j=1}^{d} z_k \cdot [X_j^{(r-2)}] \,|\, z_k \in \mathbb{Z}, \sum_{j=1}^{d} z_j \in d\mathbb{Z}\} \tag{4.29}$$

by (4.27) and (4.28) (for $[X_j^{(r-2)}]$). □

Theorem 4.5. *(a)* [Co82, 18.] *For* $s \in \{[\frac{n+2}{2}], [\frac{n+4}{2}], ..., n-1\}$

$$d^1_{s,n-1-s} : E^1_{s,n-1-s} \to E^1_{s-1,n-1-s} \quad \text{is injective,} \tag{4.30}$$

$$E^r_{s,n-1-s} = E^\infty_{s,n-1-s} = 0 \qquad \text{for } r \geq 2. \tag{4.31}$$

(b) For $t \in \{3, ..., [\frac{n+3}{2}]\}$ *and* $3 \leq r \leq t$

$$E^{r-1}_{n-t+1,t-3} \subset \frac{E^1_{n-t+1,t-3}}{d^1(E^1_{n-t+2,t-3})}. \tag{4.32}$$

(c) For $s < [\frac{n+2}{2}]$ *and* $r \geq 1$

$$E^r_{s-1,n-1-s} = 0. \tag{4.33}$$

(d) Recall the definition of $\widetilde{E}^r_{n,-1}$ *in Lemma 4.4.*

$$E^r_{n,-1} = \widetilde{E}^r_{n,-1} \quad \text{for } r \in \{1, ..., [\frac{n+3}{2}]\}, \tag{4.34}$$

$$E^\infty_{n,-1} = E^r_{n,-1} = \widetilde{E}^{[\frac{n+3}{2}]}_{n,-1} \quad \text{for } r \geq [\frac{n+3}{2}]. \tag{4.35}$$

Proof: (a) (4.31) is an immediate consequence of (4.30), because of the definition of $E^r_{s,n-1-s}$ in (4.21).

The injectivity of $d^1_{s,n-1-s}$ was proved in [Co82, 18.]. Our proof differs in a way which allows to apply it also to the proof of (4.34) in part (d). First we consider some useful sets and maps. Fix $s \in \{[\frac{n+2}{2}], [\frac{n+4}{2}], ..., n-1\}$.

$$\begin{aligned}
\mathcal{A}_1(s) &:= \{A \subset N \,|\, A \text{ thick}, |A| = s\}, \\
\mathcal{A}_2(s) &:= \{(A, j) \,|\, A \in \mathcal{A}_1(s), j \in \{1, ..., b(A)\} \\
&\qquad\qquad \text{with } k_j^A, (k_j^A + 1)_{\mathrm{mod}\, n} \in A\}, \\
\mathrm{pr}_1 &: \mathcal{A}_2(s) \to \mathcal{A}_1(s), (A, j) \mapsto A, \text{ the canonical projection.}
\end{aligned}$$

For the given s, each $A \in \mathcal{A}_1(s)$ contains a block which consists of ≥ 2 elements. Therefore $\mathrm{pr}_1 : \mathcal{A}_2(s) \to \mathcal{A}_1(s)$ is surjective.

$$\begin{aligned}
\mathcal{B}(s) &:= \{B \subset N \mid B \text{ almost thick}, |B| = s-1\}, \\
\beta_1 &: \mathcal{B}(s) \to N, \beta_1(B) := k_0 \text{ if } \{k_0, (k_0+1)_{\mathrm{mod}\, n}\} \subset N - B, \\
\beta_2 &: \mathcal{B}(s) \to N, \beta_2(B) := (\beta_1(B)+1)_{\mathrm{mod}\, n}, \\
\alpha_1 &: \mathcal{A}_2(s) \to \mathcal{B}(s), (A,j) \mapsto A - \{(k_{(j+1)_{\mathrm{mod}\, d}} - 2)_{\mathrm{mod}\, n}\}, \\
\alpha_2 &: \mathcal{A}_2(s) \to \mathcal{B}(s), (A,j) \mapsto A - \{k_j^A\}.
\end{aligned}$$

$\beta_1(B)$ and $\beta_2(B)$ are the first and last element of the unique block of $N - B$ with two elements. k_j^A is the beginning of a block of A with ≥ 2 elements, and $(k_{(j+1)_{\mathrm{mod}\, d}} - 2)_{\mathrm{mod}\, n}\}$ is the last element of this block. α_1 and α_2 are bijections, and (see the notations 3.1 for $B^{(1)}$ and $B^{(2)}$)

$$\begin{aligned}
B^{(1)} &= B \cup \{\beta_1(B)\} = \mathrm{pr}_1(\alpha_1^{-1}(B)), \\
B^{(2)} &= B \cup \{\beta_2(B)\} = \mathrm{pr}_1(\alpha_2^{-1}(B)).
\end{aligned}$$

Recall what (4.8), (4.10) and (4.12) say about $E^1_{s,n-1-s}$ and $E^1_{s-1,n-1-s}$. Both are $\mathbb{Z}$-lattices. The generators of $E^1_{s,n-1-s}$ are simply the classes $[\Delta_A \odot T_A]$, $A \in \mathcal{A}_1(s)$,

$$\begin{aligned}
E^1_{s,n-1-s} &= H_{n-1}(G_s, G_{s-1}) \\
&= \bigoplus_{A \in \mathcal{A}_1(s)} H_{n-1}(\Delta_A \odot T_A, \partial\Delta_A \odot T_A) = \bigoplus_{A \in \mathcal{A}_1(s)} \mathbb{Z} \cdot [\Delta_A \odot T_A],
\end{aligned} \tag{4.36}$$

$E^1_{s-1,n-1-s}$ splits into two parts, one from almost thick sets, the other from thick sets,

$$\begin{aligned}
E^1_{s-1,n-1-s} &= H_{n-2}(G_{s-1}, G_{s-2}) \\
&= \big(H_{n-2}(G_{s-1}, G_{s-2})\big)_{\mathcal{B}} \oplus \big(H_{n-2}(G_{s-1}, G_{s-2})\big)_{\mathcal{A}} \text{ with} \\
\big(H_{n-2}(G_{s-1}, G_{s-2})\big)_{\mathcal{B}} &:= \bigoplus_{B \in \mathcal{B}(s)} \mathbb{Z} \cdot [\Delta_B \odot T_B], \\
\big(H_{n-2}(G_{s-1}, G_{s-2})\big)_{\mathcal{A}} &:= \bigoplus_{A \in \mathcal{A}_1(s-1)} H_{s-1}(\Delta_A, \partial\Delta_A) \otimes H_{b(A)-1}(T_A) \\
&= \bigoplus_{A \in \mathcal{A}_1(s-1)} \bigoplus_{i=1}^{b(A)} \mathbb{Z} \cdot [\Delta_A \odot T_{A \cup \{(k_i^A - 1)_{\mathrm{mod}\, n}\}}].
\end{aligned} \tag{4.37}$$

We will work mainly with the part from almost thick sets. The projection to this part is called $\mathrm{pr}_{\mathcal{B}}$,

$$\mathrm{pr}_{\mathcal{B}} : E^1_{s-1,n-1-s} = H_{n-2}(G_{s-1}, G_{s-2}) \to (H_{n-2}(G_{s-1}, G_{s-2}))_{\mathcal{B}}.$$

We will prove that $\mathrm{pr}_{\mathcal{B}} \circ d^1_{s,n-1-s}$ is injective. This implies that $d^1_{s,n-1-s}$ is injective. Recall $d^1_{s,n-1-s} = j^1_{s-1,n-1-s} \circ k^1_{s,n-1-s}$, and $k^1_{s,n-1-s}$ is a boundary map. Therefore for $A \in \mathcal{A}_1(s)$

$$\begin{aligned}
d^1_{s,n-1-s}([\Delta_A \odot T_A]) &= [\partial\Delta_A \odot T_A] \\
&= \sum_{B \subset A:\, B \in \mathcal{B}(s) \cup \mathcal{A}_1(s-1)} \mathrm{sign}(B, A) \cdot [\Delta_B \odot T_A],
\end{aligned} \tag{4.38}$$

We care only about the terms for $B \subset A$ with $B \in \mathcal{B}(s)$. These sets B split into two types,

$$\{B \subset A \mid B \in \mathcal{B}(s)\} = \{B \subset A \mid A = B^{(1)}\} \cup \{B \subset A \mid A = B^{(2)}\}. \tag{4.39}$$

We claim to have on the level of chains

$$\Delta_B \odot T_A = \begin{cases} \Delta_B \odot T_B & \text{if } A = B^{(1)}, \\ (-a_{\beta_2(B)}) \cdot \Delta_B \odot T_B & \text{if } A = B^{(2)}, \beta_2(B) \neq n, \\ (-1)^{b(B)-1}(-a_n)\Delta_B \odot T_B & \text{if } A = B^{(2)}, \beta_2(B) = n. \end{cases} \tag{4.40}$$

This follows from comparison of T_A with T_B. If $B = \alpha_1(A, j)$ then $\beta_1(B) = (k_{(j+1)_{\mathrm{mod}\, n}} - 2)_{\mathrm{mod}\, n}$ is the last element of the j-th block of A. If $B = \alpha_2(A, j)$ then $\beta_2(B) = k_j^A$ is the beginning of the j-th block of A. The generating vectors of T_A and T_B from these blocks of A and B differ as follows,

$$\begin{aligned} \mathrm{pr}_B(\underline{d}_j^A) &= \mathrm{pr}_B(\underline{d}_j^B) \quad \text{if } B = \alpha_1(A, j), \\ \mathrm{pr}_B(\underline{d}_j^A) &= (-a_{\beta_2(B)}) \cdot \mathrm{pr}_B(\underline{d}_{\widetilde{j}}^B) \quad \text{if } B = \alpha_2(A, j), \end{aligned} \tag{4.41}$$

where $\widetilde{j} = j$ if $\beta_2(B) \neq n$ and where in the case $\beta_2(B) = n$

$$\begin{aligned} j = b(B) = b(A),\ \widetilde{j} = 1, \\ \mathrm{pr}_B(\underline{d}_i^A) = \mathrm{pr}_B(\underline{d}_{i+1}^B) \text{ for } i \in \{1, ..., b(B) - 1\}. \end{aligned}$$

This shows (4.40). Together (4.38), (4.39) and (4.40) give

$$\begin{aligned} &\mathrm{pr}_{\mathcal{B}}\left(d^1_{s,n-1-s}([\Delta_A \odot T_A])\right) = \sum_{B \in \alpha_1(\mathrm{pr}_1^{-1}(A))} \mathrm{sign}(B, A) \cdot [\Delta_B \odot T_B] \\ &+ \sum_{B \in \alpha_2(\mathrm{pr}_1^{-1}(A)), \beta_2(B) \neq n} \mathrm{sign}(B, A) \cdot (-a_{\beta_2(B)}) \cdot [\Delta_B \odot T_B] \\ &+ \sum_{B \in \alpha_2(\mathrm{pr}_1^{-1}(A)), \beta_2(B) = n} (-1)^{b(B)-1} \mathrm{sign}(B, A) \cdot (-a_n) \cdot [\Delta_B \odot T_B]. \end{aligned} \tag{4.42}$$

Recall from (3.23) and (3.24)

$$\begin{aligned} \mathrm{sign}(B, B^{(2)}) &= \mathrm{sign}(B, B^{(1)}) \quad \text{if } \beta_2(B) \neq n, \\ 1 = \mathrm{sign}(B, B^{(2)}) &= (-1)^{|B|} \mathrm{sign}(B, B^{(1)}) \quad \text{if } \beta_2(B) = n, \\ \text{and } (-1)^{b(B)-1+|B|} &= (-1)^n. \end{aligned}$$

In view of these signs and (4.42), an arbitrary linear combination $\sum_{A \in \mathcal{A}_1(s)} z_A \cdot [\Delta_A \odot T_A]$, $z_A \in \mathbb{Z}$, is mapped by $\mathrm{pr}_{\mathcal{B}} \circ d^1_{s,n-1-s}$ to

$$\mathrm{pr}_{\mathcal{B}}\big(d^1_{s,n-1-s}\Big(\sum_{A \in \mathcal{A}_1(s)} z_A \cdot [\Delta_A \odot T_A]\Big)\big) \tag{4.43}$$

$$\begin{aligned} = &\sum_{B \in \mathcal{B}(s), \beta_2(B) \neq n} \mathrm{sign}(B, B^{(1)})\big(z_{B^{(1)}} - a_{\beta_2(B)} z_{B^{(2)}}\big)[\Delta_B \odot T_B] \\ + &\sum_{B \in \mathcal{B}(s), \beta_2(B) = n} \mathrm{sign}(B, B^{(1)})\big(z_{B^{(1)}} - (-1)^n a_n z_{B^{(2)}}\big)[\Delta_B \odot T_B]. \end{aligned}$$

The following Claim 1 will be useful here and in the proof of (4.34) in part (d).

Claim 1: *Fix any $B \in \mathcal{B}(s)$. A sequence $(B_i)_{i\in N}$ of elements $B_i \in \mathcal{B}(s)$ with $B = B_n$ and with the following property exists,*

$$B^{(1)}_{(i+1)_{\mathrm{mod}\, n}} = B^{(2)}_i \quad \text{for } i \in N. \tag{4.44}$$

Additionally,

$$\text{the map} \quad N \to N,\ i \mapsto \beta_2(B_i), \quad \text{is a bijection.} \tag{4.45}$$

Proof of Claim 1: We will describe such a sequence from a point of view which will make its existence clear. Suppose we have such a sequence $(B_i)_{i\in N}$. Compare the set of gaps of $B^{(2)}_{(i-1)_{\mathrm{mod}\, n}} = B^{(1)}_i$ with the set of gaps of $B^{(2)}_i = B^{(1)}_{(i+1)_{\mathrm{mod}\, n}}$. These sets almost coincide. Only the gap $\{\beta_2(B_i)\}$ of $B^{(1)}_i$ is shifted one position to the left to the gap $\{\beta_1(B_i)\}$ of $B^{(2)}_i$. Starting from the set of gaps of $B^{(2)}_n = B^{(1)}_1$, one shifts in n steps all gaps to the left, so that the final position of each gap is the original position of the gap left of it. One has to take care that the gaps stay always apart from one another. It is clear that starting from the set of gaps of $B^{(2)}_n$, one can find such n steps. Also (4.45) is clear. This finishes the proof of Claim 1. (□)

Consider now a linear combination $\sum_{A\in\mathcal{A}_1(s)} z_A \cdot [\Delta_A \odot T_A]$, $z_A \in \mathbb{Z}$, which is mapped by $\mathrm{pr}_{\mathcal{B}} \circ d^1_{s,n-1-s}$ to 0. Choose any $A \in \mathcal{A}_1(s)$ and choose $B \in \mathcal{B}(s)$ with $B^{(2)} = A$. Claim 1 provides a sequence $(B_i)_{i\in N}$ with $B_n = B$, $B^{(2)}_n = A$, (4.44) and (4.45). Then (4.43) gives the relations

$$z_{B^{(2)}_{(i-1)_{\mathrm{mod}\, n}}} = z_{B^{(1)}_i} = \begin{cases} a_{\beta_2(B_i)} z_{B^{(2)}_i} & \text{if } \beta_2(B_i) \neq n, \\ (-1)^n a_n z_{B^{(2)}_i} & \text{if } \beta_2(B_i) = n. \end{cases} \tag{4.46}$$

Together these relations and (4.45) imply especially

$$z_A = \underline{a}_{n+1} \cdot z_A, \quad \text{so } (-1)^n d \cdot z_A = 0, \quad \text{so } z_A = 0.$$

Therefore $\mathrm{pr}_{\mathcal{B}} \circ d^1_{s,n-1-s}$ is injective, and thus also $d^1_{s,n-1-s}$ is injective. This finishes the proof of part (a).

(b) The definition (4.21) of $E^{r-1}_{n-t+1,t-3}$ for $r \geq 3$ gives

$$E^{r-1}_{n-t+1,t-3} \quad \subset \quad \frac{E^{r-2}_{n-t+1,t-3}}{d^{r-2}(E^{r-2}_{n-t+r-1,t-r})}.$$

If $r = 3$, this is (4.32). If $r \geq 4$, (4.31) gives $E^{r-2}_{n-t+r-1,t-r} = 0$, so $E^{r-1}_{n-t+1,t-3} \subset E^{r-2}_{n-t+1,t-3}$. Induction gives (4.32).

(c) $E^1_{s-1,n-1-s} = H_{n-2}(G_{s-1}, G_{s-2})$ has by (4.8), (4.10) and (4.12) only contributions from sets $A \subset N$ with $|A| = s-1$ and $b(A) \in \{n-|A|, n-1-|A|\}$. These are thick or almost thick sets. By Lemma 3.2, for $s < [\frac{n+2}{2}]$, there are no thick or almost thick sets with $|A| = s-1$. Therefore then $E^1_{s-1,n-1-s} = 0$ and $E^r_{s-1,n-1-s} = 0$ for $r \geq 1$.

(d) By (4.23), for $r \geq 2$

$$E^r_{n,-1} = \ker(d^{r-1}_{n,-1} : E^{r-1}_{n,-1} \to E^{r-1}_{n-r+1,r-3}) \subset E^{r-1}_{n,-1} \subset E^1_{n,-1}. \tag{4.47}$$

For $r \geq [\frac{n+3}{2}] + 1$, $E^{r-1}_{n-r+1,r-3} = 0$ by part (c). Therefore then $E^r_{n,-1} = E^{r-1}_{n,-1}$. Inductively we obtain

$$E^\infty_{n,-1} = E^r_{n,-1} = E^{[\frac{n+3}{2}]}_{n,-1} \quad \text{for } r \geq [\frac{n+3}{2}].$$

It remains to prove (4.34). We will prove it by induction in r. By definition $\widetilde{E}^1_{n,-1} = E^1_{n,-1}$. First we treat the special case $r = 2$:

$$\begin{aligned}
E^2_{n,-1} &= \ker(d^1 : E^1_{n,-1} \to E^1_{n-1,-1}),\\
d^1([X^{(0)}_j]) &= d^1([\Delta_N \odot \{\underline{e}(\underline{p}_j)\}]) = [\partial\Delta_N \odot \{\underline{e}(\underline{p}_j)\}]\\
&= \sum_{B\subset N:\,|B|=n-1} \operatorname{sign}(B,N)\cdot[\Delta_B \odot \{\underline{e}(\underline{p}_j)\}]\\
&= \sum_{B\subset N:\,|B|=n-1} \operatorname{sign}(B,N)\cdot[\Delta_B \odot \{\underline{e}(\underline{p}_1)\}]\\
&\in E^1_{n-1,-1} = H_{n-2}(G_{n-1}, G_{n-2})\\
&= \bigoplus_{i=1}^n \mathbb{Z}\cdot[\Delta_{N-\{i\}} \odot \{\underline{e}(\underline{p}_1)\}] \cong \mathbb{Z}^n.
\end{aligned}$$

The last equality follows from (4.8) and (4.10). Obviously $d^1([X^{(0)}_j])$ is independent of j and is $\neq 0$. Therefore

$$\begin{aligned}
E^2_{n,-1} &= \ker(d^1 : E^1_{n,-1} \to E^1_{n-1,-1})\\
&= \{\sum_{j=1}^d z_j\cdot[X^{(0)}_j] \,|\, z_j\in\mathbb{Z}, \sum_{j=1}^d z_j = 0\} = \widetilde{E}^2_{n,-1}.
\end{aligned}$$

Now we suppose $r \in \{3, ..., [\frac{n+3}{2}]\}$ and (induction hypothesis) $E^{r-1}_{n,-1} = \widetilde{E}^{r-1}_{n,-1}$. This induction hypothesis and (4.26) give

$$E^{r-1}_{n,-1} = \widetilde{E}^{r-1}_{n,-1} = \bigoplus_{j=1}^{d-1}\mathbb{Z}\cdot[X^{(r-2)}_j] = \sum_{j=1}^{d}\mathbb{Z}\cdot[X^{(r-2)}_j] \subset E^1_{n,-1}. \tag{4.48}$$

We have to control $d^{r-1}_{n,-1}([X^{(r-2)}_j])$. Recall four points:

(i) $d^{r-1}_{n,-1} = j^{r-1}_{n-1,-1}\circ k^{r-1}_{n,-1}$.

(ii) $k^{r-1}_{n,-1}$ is a boundary map with image in

$$A^{r-1}_{n-1,-1} = (i^1)^{r-2}(A^1_{n-r+1,r-3}) = \operatorname{Im}(H_{n-2}(G_{n-r+1}) \to H_{n-2}(G_{n-1})).$$

(iii) $j^{r-1}_{n-1,-1}$ maps this space to $E^{r-1}_{n-r+1,r-3}$, which satisfies because of (4.32) (for $r = t$)

$$E^{r-1}_{n-r+1,r-3} \subset \frac{E^1_{n-r+1,r-3}}{d^1(E^1_{n-r+2,r-3})} = \frac{H_{n-2}(G_{n-r+1}, G_{n-r})}{d^1(E^1_{n-r+2,r-3})}.$$

(iv) (3.16) in Theorem 3.3 gives $\partial X^{(r-2)}_j = \partial_1 R^{(r-2)}_j$, and this is a chain in G_{n-r+1}, which fits.

These four points imply

$$d^{r-1}_{n,-1}([X^{(r-2)}_j] = [\partial_1 R^{(r-2)}_j] \in \frac{H_{n-2}(G_{n-r+1}, G_{n-r})}{d^1(E^1_{n-r+2,r-3})}. \tag{4.49}$$

We will show the following claim.

Claim 2: *The class $[\partial_1 R^{(r-2)}_j]$ in the quotient in (4.49) of the chain $\partial_1 R^{(r-2)}_j$ is independent of j, and for $m\in\mathbb{Z}$*

$$m\cdot[\partial_1 R^{(r-2)}_j] = 0 \iff d|m. \tag{4.50}$$

Claim 2 implies

$$\begin{aligned} E^r_{n,-1} &= \ker(d^{r-1}_{n,-1} : E^{r-1}_{n,-1} \to E^{r-1}_{n-r+1,r-3}) \\ &= \{\sum_{j=1}^d z_j \cdot [X_j^{(r-2)}] \mid z_j \in \mathbb{Z}, \sum_{j=1}^d z_j \in d\mathbb{Z}\} \stackrel{(4.29)}{=} \widetilde{E}^r_{n,-1}. \end{aligned}$$

It remains to prove Claim 2.

The chain $\partial_1 R_j^{(r-2)}$ was already studied in the proof of Theorem 3.3. Formula (3.22) showed contributions from pairs (B, A) of sets $B \subset A \subset N$ with $|B| = |A| - 1$, $|A| = n - r + 2$ and A thick. A priori B could be one of three types I, II and III. But a set B of type III gives nothing. A set B of type II is almost thick and gives either nothing or the contribution in (3.25) (up to a sign) which is independent of j. A set B of type I is thick and gives the contribution in (3.26), which we call now $\mathrm{Cont}(B, \partial_1 R_j^{(r-2)})$.

First we will show that $\mathrm{Cont}(B, \partial_1 R_j^{(r-2)})$ is a relative cycle, so that it gives a class

$$[\mathrm{Cont}(B, \partial_1 R_j^{(r-2)})] \in H_{n-2}(G_{n-r+1}, G_{n-r}). \tag{4.51}$$

Then we will show that this class is independent of j.

$\partial_1 \left(\mathrm{Cont}(B, \partial_1 R_j^{(r-2)})\right)$ contains $\partial\Delta_B$, so it is in G_{n-r}, and we can ignore it. For the signs in (3.26), recall (3.27). Then

$$\begin{aligned} &\partial_2 \left(\mathrm{Cont}(B, \partial_1 R_j^{(r-2)})\right) \\ = &\sum_{i=1}^{b(B)} \mathrm{sign}(B)(-1)^{|B|+i-1} \cdot \Delta_B \odot \left(\underline{p}_j + d^{-1}\partial C(\underline{c}_{k_1^B}, ..., \widehat{\underline{c}_{k_i^B}}, ..., \underline{c}_{k^B_{b(B)}})\right). \end{aligned}$$

This is equal to 0, because for each pair $(i, l) \in \{1, ..., b(B)\}^2$ with $i < l$, the term $\underline{p}_j + d^{-1}C(\underline{c}_{k_1^B}, ..., \widehat{\underline{c}_{k_i^B}}, ..., \widehat{\underline{c}_{k_l^B}}, ..., \underline{c}_{k^B_{b(B)}})$ and the term $\underline{p}_{j+1} + d^{-1}C(\underline{c}_{k_1^B}, ..., \widehat{\underline{c}_{k_i^B}}, ..., \widehat{\underline{c}_{k_l^B}}, ..., \underline{c}_{k^B_{b(B)}})$ turn up twice and with different signs. Therefore $\mathrm{Cont}(B, \partial_1 R_j^{(r-2)})$ is a relative cycle. In the proof of Theorem 3.3, we found

$$\begin{aligned} &- \mathrm{Cont}(B, \partial_1 R_j^{(r-2)}) + \mathrm{Cont}(B, \partial_1 R_{j+1}^{(r-2)}) \\ &= \mathrm{Cont}(B, \partial_2 R_j^{(r-1)}) \\ &= \partial_2(\mathrm{Cont}(B, R_j^{(r-1)})) \\ &= \partial(\mathrm{Cont}(B, R_j^{(r-1)})) - \partial_1(\mathrm{Cont}(B, R_j^{(r-1)})). \end{aligned}$$

In the last difference, the first term is a boundary, and the second term is in G_{n-r}. Therefore the class in $H_{n-2}(G_{n-r+1}, G_{n-r})$ of this difference is 0. Therefore the class in (4.51) is independent of j.

We see that all contributions to the class $[\partial_1 R_j^{(r-2)}] \in H_{n-2}(G_{n-r+1}, G_{n-r})$ are independent of j, so this class is independent of j. This implies the first statement in Claim 2.

By (4.28) $0 = \sum_{j=1}^d [X_j^{(r-2)}] \in E^1_{n,-1}$. Thus

$$0 = \sum_{j=1}^d d^{r-1}_{n,-1}([X_j^{(r-2)}]) = d \cdot [\partial_1 R_j^{(r-2)}] \quad \text{for any } j \in \{1, ..., d\}$$

in the quotient in (4.49). This is $\Leftarrow$ of (4.50).

It remains to prove $\Rightarrow$ of (4.50). Define $s := n - r + 2$. Then $r \in \{3, ..., [\frac{n+3}{2}]\}$ implies $s \in \{[\frac{n+2}{2}], ..., n - 1\}$. So part (a) and its proof apply. The map $\mathrm{pr}_{\mathcal{B}} \circ d^1_{n-r+2,r-3}$ is injective, and formula (4.43) gives its image in $H_{n-2}(G_{n-r+1}, G_{n-r})_{\mathcal{B}}$. For $\Rightarrow$ of (4.50), it is sufficient to show for $m \in \mathbb{Z}$

$$m \cdot \mathrm{pr}_{\mathcal{B}}([\partial_1 R_j^{(r-2)}]) \in \mathrm{pr}_{\mathcal{B}}(\mathrm{Im}(d^1_{n-r+2,r-3})) \quad \Rightarrow \quad d|m, \tag{4.52}$$

here $[\partial_1 R_j^{(r-2)}]$ denotes the class in $H_{n-2}(G_{n-r+1}, G_{n-r})$ (not the class in the quotient in (4.49)).

For $\mathrm{pr}_{\mathcal{B}}([\partial_1 R_j^{(r-2)}])$, we need only the contributions of sets $B \in \mathcal{B}(n-r+2)$. By the proof of Theorem 3.3, the contribution of such a set B is 0 if $\beta_2(B) \neq n$, and it is given by (3.25) if $\beta_2(B) = n$. We obtain

$$\mathrm{pr}_{\mathcal{B}}([\partial_1 R_j^{(r-2)}]) = \sum_{B\in\mathcal{B}(n-r+2):\, \beta_2(B)=n} (\pm 1)\underline{a}_1 \underline{a}_{k_2^B} ... \underline{a}_{k^B_{b(B)}} \cdot [\Delta_B \odot T_B]\,.$$

Suppose that the left-hand side of (4.52) holds for some $m \in \mathbb{Z}$. Write

$$\lambda_B := m \cdot (\pm 1)\underline{a}_1 \underline{a}_{k_2^B} ... \underline{a}_{k^B_{b(B)}}\,.$$

Then we have a linear combination $\sum_{A\in\mathcal{A}_1(n-r+2)} z_A \cdot [\Delta_A \odot T_A]$ with

$$\begin{aligned} m \cdot \mathrm{pr}_{\mathcal{B}}([\partial_1 R_j^{(r-2)}]) &= \sum_{B\in\mathcal{B}(n-r+2):\, \beta_2(B)=n} \lambda_B \cdot [\Delta_B \odot T_B] \\ &= \mathrm{pr}_{\mathcal{B}}\left(d^1_{n-r+2,r-3}\Big(\sum_{A\in\mathcal{A}_1(n-r+2)} z_A \cdot [\Delta_A \odot T_A]\Big)\right). \end{aligned} \tag{4.53}$$

Choose a set $B \in \mathcal{B}(n-r+2)$ with $\beta_2(B) = n$. Claim 1 provides a sequence $(B_i)_{i\in N}$ of elements $B_i \in \mathcal{B}(n-r+2)$ with $B = B_n$, (4.44) and (4.45). Then (4.53) and (4.43) give the first line of (4.46), so

$$z_{B^{(2)}} = \Big(\prod_{i=1}^{n-1} a_i\Big) \cdot z_{B^{(1)}}.$$

By (4.43), the coefficient λ_B is

$$\begin{aligned} \lambda_B &= \mathrm{sign}(B, B^{(1)}) \cdot \Big(z_{B^{(1)}} - (-1)^n a_n z_{B^{(2)}} \Big) \\ &= \mathrm{sign}(B, B^{(1)}) \cdot z_{B^{(1)}} \cdot (1 - \underline{a}_{n+1}) \\ &= \mathrm{sign}(B, B^{(1)}) \cdot z_{B^{(1)}} \cdot (-1)^{n+1} d. \end{aligned} \tag{4.54}$$

Therefore $\lambda_B \in d\mathbb{Z}$. Observe $\gcd(d, \underline{a}_1 \underline{a}_{k_2^B} ... \underline{a}_{k^B_{b(B)}}) = 1$. Therefore, $d|m$. (4.52) is proved. This finishes the proof of part (d). □

Theorem 4.6. *The homology group $H_{n-1}(G,\mathbb{Z}) = H_{n-1}(G)$ is a $\mathbb{Z}$-lattice of rank*

$$\mu = (-1)^n \underline{a}_{n+1} = d + (-1)^n.$$

A $\mathbb{Z}$-basis of it is

$$\begin{array}{ll} [X_1^{(\frac{n}{2})}], ..., [X_{d-1}^{(\frac{n}{2})}], [\Delta_{A_{od}} \odot T_{A_{od}}], [\Delta_{A_{ev}} \odot T_{A_{ev}}], & \text{if } n \text{ is even,} \\ [X_1^{(\frac{n+1}{2})}], ..., [X_{d-1}^{(\frac{n+1}{2})}], & \text{if } n \text{ is odd.} \end{array} \tag{4.55}$$

Together with $[X_d^{([\frac{n+1}{2}])}]$, these elements satisfy the relation (3.18) if n is odd and the relation (3.19) if n is even.

Proof: By Lemma 3.2, in the case n even, $[\Delta_{A_{od}} \odot T_{A_{od}}]$ and $[\Delta_{A_{ev}} \odot T_{A_{ev}}]$ are homology classes in $H_{n-1}(G)$. By Theorem 3.3, $[X_1^{([\frac{n+1}{2}])}], ..., [X_d^{([\frac{n+1}{2}])}]$ are homology classes in $H_{n-1}(G)$, the relation (3.18) holds if n is odd, and the relation (3.19) holds if n is even. We have to show that the elements in (4.55) form a $\mathbb{Z}$-basis of $H_{n-1}(G)$.

First we consider even n. By Lemma 4.3 (b) and (c) and Theorem 4.5 (a), $E^\infty_{s,n-1-s} = 0$ for all s except $s = n$ and $s = \frac{n}{2}$. Therefore $F^s_{n-1} = 0$ for all $s < \frac{n}{2}$, and by Lemma 4.3 (c)

$$\begin{aligned} F^{\frac{n}{2}}_{n-1} &= F^{\frac{n}{2}+1}_{n-1} = ... = F^{n-1}_{n-1} = \mathrm{Im}(H_{n-1}(G_{\frac{n}{2}}) \to H_{n-1}(G)) \\ &\cong E^\infty_{\frac{n}{2},\frac{n}{2}-1} = E^1_{\frac{n}{2},\frac{n}{2}-1} = H_{n-1}(G_{\frac{n}{2}}, G_{\frac{n}{2}-1}) \cong \mathbb{Z}^2 \end{aligned} \tag{4.56}$$

with generators the classes in $H_{n-1}(G)$ respectively in $H_{n-1}(G_{\frac{n}{2}}, G_{\frac{n}{2}-1})$ of the spheres

$$\Delta_{A_{od}} \odot T_{A_{od}} \subset \mathbb{C}^{A_{od}}$$

and $\Delta_{A_{ev}} \odot T_{A_{ev}} \subset \mathbb{C}^{A_{ev}}$. By Theorem 4.5 (d) and Lemma 4.4

$$E^\infty_{n,-1} = \widetilde{E}^{\frac{n+2}{2}}_{n,-1} = \bigoplus_{j=1}^{d-1} \mathbb{Z} \cdot [X_j^{(\frac{n}{2})}] \subset E^1_{n,-1} = H_{n-1}(G, G_{n-1}), \tag{4.57}$$

$E^\infty_{n,-1}$ is a $\mathbb{Z}$-lattice of rank $d-1$, and

$$E^\infty_{n,-1} \cong \frac{F^n_{n-1}}{F^{\frac{n}{2}}_{n-1}} = \frac{H_{n-1}(G)}{\mathrm{Im}(H_{n-1}(G_{\frac{n}{2}}) \to H_{n-1}(G)))}. \tag{4.58}$$

Therefore $H_{n-1}(G)$ is a $\mathbb{Z}$-lattice of rank $d+1$ with $\mathbb{Z}$-basis as in (4.55), and $F^{\frac{n}{2}}_{n-1}$ is a primitive sublattice of rank 2 with generators the classes of the spheres $\Delta_{A_{od}} \odot T_{A_{od}}$ and $\Delta_{A_{ev}} \odot T_{A_{ev}}$.

Now we consider odd n. By Lemma 4.3 (b) and Theorem 4.5 (a), $E^\infty_{s,n-1-s} = 0$ for all $s \neq n$. Therefore $F^s_{n-1} = 0$ for all $s \neq n$ and

$$H_{n-1}(G) = F^n_{n-1} \cong E^\infty_{n,-1}. \tag{4.59}$$

By Theorem 4.5 (d) and Lemma 4.4

$$E^\infty_{n,-1} = \widetilde{E}^{\frac{n+3}{2}}_{n,-1} = \bigoplus_{j=1}^{d-1} \mathbb{Z} \cdot [X_j^{(\frac{n+1}{2})}] \subset E^1_{n,-1} = H_{n-1}(G, G_{n-1}), \tag{4.60}$$

and $E^\infty_{n,-1}$ is a $\mathbb{Z}$-lattice of rank $d-1$. Therefore $H_{n-1}(G)$ is a $\mathbb{Z}$-lattice of rank $d-1$ with $\mathbb{Z}$-basis as in (4.55). □

Remarks 4.7. In the proof of part (d) of Theorem 4.5, we considered the contribution $\mathrm{Cont}(B, \partial_1 R_j^{(r-2)})$ of a set $B \in \mathcal{A}_1(n-r+1)$ (B is thick with $|B| = n-r+1$) to the chain $\partial_1 R_j^{(r-2)}$. We showed that it is a relative cycle in $H_{n-2}(G_{n-r+1}, G_{n-r})$, and we saw that it is independent of j. We did not make precise which cycle it is, as we did not need this. Now we will say which cycle it is, but leave the proof to the reader:

$$\begin{aligned} & [\mathrm{Cont}(B, \partial_1 R_j^{(r-2)})] \qquad (4.61) \\ = \ & \varepsilon_1 \cdot [\Delta_B \odot \underline{e}\big(d^{-1}C(\underline{c}_{k_2^B} - \underline{c}_{k_1^B}, ..., \underline{c}_{k_{b(B)}^B} - \underline{c}_{k_{b(B)-1}^B})\big)] \\ = \ & \varepsilon_2 \cdot [\Delta_B \odot \underline{e}\big(C(\underline{b}_{k_2^B} - \underline{b}_{k_1^B}, ..., \underline{b}_{k_{b(B)}^B} - \underline{b}_{k_{b(B)-1}^B})\big)]. \\ = \ & \varepsilon_2 \cdot \underline{a}_{k_1^B}\underline{a}_{k_2^B}...\underline{a}_{k_{b(B)-1}^B} \cdot [\Delta_B \odot \underline{e}\big(C(\underline{d}_1^B, ..., \underline{d}_{b(B)-1}^B)\big)]. \\ & \text{with } \varepsilon_1 = \mathrm{sign}(B)(-1)^{|B|},\ \varepsilon_2 = \varepsilon_1 \cdot (-1)^{n(b(B)-1)}. \end{aligned}$$

5. Proof of the main result

Throughout this section we fix a cycle type singularity $f(x_1, ..., x_n)$ as in (1.2) with $n \geq 2$ and $a_1, ..., a_n \in \mathbb{N}$ with (1.3), as in the sections 3 and 4. We want to prove the main result of this paper, Theorem 1.3.

The deformation lemma in [Co82, 3.] implies that the inclusion map $i_g : G \hookrightarrow \overline{F_0}$ induces an epimorphism $(i_g)_* : H_{n-1}(G, \mathbb{Z}) \to H_{n-1}(\overline{F_0}, \mathbb{Z})$. Both groups are $\mathbb{Z}$-lattices of rank μ. This holds by Theorem 4.6 for $H_{n-1}(G, \mathbb{Z})$ and by [Mi68] for $H_{n-1}(\overline{F_0}, \mathbb{Z})$. Therefore the epimorphism is an isomorphism:

$$(i_g)_* : H_{n-1}(G, \mathbb{Z}) \to H_{n-1}(\overline{F_0}, \mathbb{Z}).$$

It remains to determine the monodromy on $H_{n-1}(G, \mathbb{Z})$. We call this monodromy h_{mon}. First we give the weights $(w_1, ..., w_n)$ of the quasihomogeneous polynomial f (see e.g. Lemma 4.1 in [HZ19]):

$$\begin{aligned} w_j &= \frac{v_j}{d} \in \mathbb{Q} \cap (0,1) \quad \text{for } j \in N \quad \text{with} \\ d &= \prod_{j=1}^n a_j - (-1)^n = \mu - (-1)^n, \\ v_j &= \sum_{l=1}^n (-1)^{l-1} \prod_{k=j+l}^{j+n-1} a_{(k)_{\mathrm{mod}\, n}}, \text{ with} \qquad (5.1) \\ & a_j v_j + v_{(j+1)_{\mathrm{mod}\, n}} = d. \end{aligned}$$

One sees easily $\gcd(d, v_1) = ... = \gcd(d, v_n)$. We defined already $b := d/\gcd(v_1, d)$. In the case of a quasihomogeneous singularity, one can give explicitly a diffeomorphism $\Phi_{mon} : \mathbb{C}^n \to \mathbb{C}^n$ of the Milnor fibration which induces the monodromy on the homology of the Milnor fiber and on $H_{n-1}(G, \mathbb{Z})$. It looks as follows.

$$\Phi_{mon} : \mathbb{C}^n \to \mathbb{C}^n, \quad (z_1, .., z_n) \mapsto (e^{2\pi i w_1} z_1, ..., e^{2\pi i w_n} z_n).$$

It maps the point $\underline{e}(\underline{p}_j)$ to the point $\underline{e}(\underline{p}_{(j+v_1)_{\mathrm{mod}\, d}})$, because of $a_j v_j + v_{(j+1)_{\mathrm{mod}\, n}} = d$. Therefore it maps the chain $R_j^{(k)}$ to the chain $R_{(j+v_1)_{\mathrm{mod}\, d}}^{(k)}$ and the chain $X_j^{(k)}$ to the chain $X_{(j+v_1)_{\mathrm{mod}\, d}}^{(k)}$. In the case of even n, it maps the spheres $\Delta_{A_{od}} \odot T_{A_{od}} \subset \mathbb{C}^{A_{od}}$ and $\Delta_{A_{ev}} \odot T_{A_{ev}} \subset \mathbb{C}^{A_{ev}}$ to themselves.

Now we can apply the results of section 2. In order to make the relation to these results transparent, we introduce the following notations,

$$\delta_j \quad := \quad [X_j^{([\frac{n+1}{2}])}] \in H_{n-1}(G, \mathbb{Z}) \quad \text{for } j \in \{1, ..., d\},$$

for odd n or even n, and

$$\begin{aligned} \gamma &:= [\Delta_{A_{od}} \odot T_{A_{od}}] + (-1)^{\frac{n}{2}} a_1 a_3 ... a_{n-1} \cdot [\Delta_{A_{ev}} \odot T_{A_{ev}}] \in H_{n-1}(G,\mathbb{Z}), \\ \beta &:= [\Delta_{A_{ev}} \odot T_{A_{ev}}] \in H_{n-1}(G,\mathbb{Z}), \\ c &:= (-1)^{(\frac{n}{2}+2)(\frac{n}{2}+1)\frac{1}{2}} \cdot \underline{a}_1 \underline{a}_3 ... \underline{a}_{n-1} \in \mathbb{Z}, \end{aligned}$$

for even n. Then

$$\begin{aligned} H_{n-1}(G,\mathbb{Z}) &= \bigoplus_{j=1}^{d-1} \mathbb{Z}\cdot\delta_j \oplus \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \mathbb{Z}\cdot\gamma \oplus \mathbb{Z}\cdot\beta & \text{if } n \text{ is even,} \end{cases} \\ h_{mon}(\delta_j) &= \delta_{(j+v_1)\bmod d}, \\ h_{mon}(\gamma) &= \gamma, \quad h_{mon}(\beta) = \beta, \quad \text{if } n \text{ is even,} \\ \sum_{j=1}^{d} \delta_j &= \begin{cases} 0 & \text{if } n \text{ is odd (because of (3.18)),} \\ c\cdot\gamma & \text{if } n \text{ is even (because of (3.19)).} \end{cases} \end{aligned}$$

The proof of Lemma 2.3 can be adapted to show for odd n

$$(H_{n-1}(G,\mathbb{Z})), h_{mon}) \cong (\gcd(v_1,d)-1)\mathrm{Or}(t^b-1) \oplus \mathrm{Or}(\frac{t^b-1}{t-1}).$$

For even n, we see with Definition 2.1, Lemma 2.3 and Lemma 2.4

$$\begin{aligned} & (H_{n-1}(G,\mathbb{Z}), h_{mon}) \\ \overset{\text{Definition 2.1}}{\cong} & \big(H^{(d,c)}, (h^{(d,c)})^{v_1}\big) \oplus \mathrm{Or}(t-1) \\ \overset{\text{Lemma 2.3}}{\cong} & (\gcd(v_1,d)-1)\mathrm{Or}(t^b-1) \oplus \mathrm{Lo}^{(b,c)} \oplus \mathrm{Or}(t-1) \\ \overset{\text{Lemma 2.4}}{\cong} & \gcd(v_1,d)\mathrm{Or}(t^b-1) \oplus \mathrm{Or}(t-1), \end{aligned}$$

the last isomorphism uses $\gcd(b,c)=1$ and (2.3). This finishes the proof of Theorem 1.3.

Remarks 5.1. Cooper's paper [Co82] studied the integral monodromy of the cycle type singularities. It is split into 22 sections. The sections 1 und 2 are an introduction and a discussion of a degenerate case. The sections 3 to 14 are devoted to the deformation lemma in section 3 in [Co82] which yields that $(i_g)_*$ is surjective.

Section 15 introduces the deformed simplices Δ_A, the tori T_A, the chains $\Delta_A \odot T_A$, the blocks of sets $A \subset N$, and it gives most of the statements of our Lemma 3.2.

Section 16 states formulas which are close to our formula (4.40).

At the end of section 17, a formula which is close to our formula (4.43) is derived. Section 17 also defines the subsets G_s and thick sets A and states a part of our formulas (4.7)–(4.13) for the relative homology groups.

Section 18 first introduces the spectral sequence $(E^r_{s,t}, d^r_{s,t})$ of the filtration

$$G = G_n \supset G_{n-1} \supset ... \supset G_1 \supset G_0 = \emptyset.$$

But it gives neither a reference, nor the properties in our Theorem 4.2. Then it states and proves the injectivity of the maps $d^1_{s,n-1-s}$ in our Theorem 4.5 (a). Though it does not use our Claim 1, but uses a specific sequence of $n\cdot(n-s)$ elements $B_i \in \mathcal{B}(s)$. Claim 1 is more efficient and can be used also for the proof of our Theorem 4.5 (d).

The lemma in Section 19 is crucial, but not correct. It claims correctly $[X_j^{(1)}] \in E^2_{n,-1}$, but wrongly $[X_j^{(2)}] \in E^\infty_{n,-1}$. By Theorem 4.5 (d), this is true only if $n \in \{3,4\}$. The wrong argument is in the proof of the lemma at the end of section 19: A part of the boundary is missed. This makes Cooper believe that $X_j^{(2)}$ is always a cycle. He constructs and sees only the beginning for $k \in \{0, 1, 2\}$ of the sequence $(X_j^{(k)})_{k\in\{1,...,[\frac{n+1}{2}]\}}$ of chains.

The lemma in section 20 has some similarity with our Claim 2 (in the proof of our Theorem 4.5 (d)). But it is too special.

The lemma in section 21 concludes wrongly that the classes $[X_j^{(2)}]_{j=1,...,d-1}$ form a $\mathbb{Z}$-basis of $E^\infty_{n,-1}$. This is true only for $n \in \{3, 4\}$. It builds on the lemma in section 19. The wrong statements on the elements and generators of $E^\infty_{n,-1}$ in the sections 19 and 21 form the first serious mistake.

The final section 22 makes the second serious mistake, in the case of even n. It states correctly that for even n $E^\infty_{s,n-1-s} \neq 0$ only if $s \in \{\frac{n}{2}, n\}$. And it derives correctly

$$(E^\infty_{\frac{n}{2},\frac{n}{2}-1}, h_{mon}) \cong 2\mathrm{Or}(t-1)$$

and

$$(E^\infty_{n,-1}, h_{mon}) \cong (\gcd(v_1, d) - 1)\mathrm{Or}(t^b - 1) \oplus \mathrm{Or}\Big(\frac{t^b - 1}{t-1}\Big). \tag{5.2}$$

But then it supposes wrongly that the quotients $E^\infty_{\frac{n}{2},\frac{n}{2}-1}$ and $E^\infty_{n,-1}$ of the filtration $F^\bullet_{n-1}$ of $H_{n-1}(G)$ lift to a splitting of $H_{n-1}(G)$ which is invariant by the monodromy. This would give a non-standard decomposition of $(H_{n-1}(G), h_{mon})$ into Orlik blocks (and this is what Cooper claims to have for even n).

But Cooper's paper contains most of the ingredients which we need for the proof of Orlik's conjecture. We corrected the two mistakes, we were more careful with the signs and proofs, we introduced more notations, we had the full sequence of chains $(X_j^{(k)})_{k\in\{1,...,[\frac{n+1}{2}]\}}$, and we had the algebraic statements in our section 2. But Cooper's paper [Co82] was an indispensable basis on which we could build.

Remarks 5.2. Orlik and Randell study in [OR77] mainly the integral monodromy of the chain type singularities. But section 3 says also something about the cycle type singularities. In (3.4.1) they make a conjecture on the integral monodromy of the cycle type singularities which would imply Orlik's conjecture. Their ansatz for a proof is very different from [Co82]. They point themselves to a gap in this ansatz.

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