

ON THE GROWTH BEHAVIOUR OF HIRONAKA QUOTIENTS

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ABSTRACT. We consider a finite analytic morphism $\phi = (f, g) : (X, p) \rightarrow (\mathbb{C}^2, 0)$ where (X, p) is a complex analytic irreducible surface germ and f and g are complex analytic function germs. Let $\pi : (Y, E_Y) \rightarrow (X, p)$ be a good resolution of ϕ with exceptional divisor $E_Y = \pi^{-1}(p)$. We denote $G(Y)$ the dual graph of the resolution π . We study the behaviour of the Hironaka quotients of (f, g) associated to the vertices of $G(Y)$. We show that there exists maximal oriented arcs in $G(Y)$ along which the Hironaka quotients of (f, g) strictly increase and they are constant on the connected components of the closure of the complement of the union of the maximal oriented arcs.

1. INTRODUCTION

Let $\phi = (f, g) : (X, p) \rightarrow (\mathbb{C}^2, 0)$ be a finite analytic morphism which is defined on a complex analytic surface germ (X, p) by two complex analytic function germs f and g .

We say that $\pi : (Y, E_Y) \rightarrow (X, p)$ is a *good resolution of ϕ* if it is a resolution of the singularity (X, p) in which the total transform $E_Y^\dagger = ((f, g) \circ \pi)^{-1}(0)$ is a normal crossings divisor and such that the irreducible components of the exceptional divisor $E_Y = \pi^{-1}(p)$ are non-singular. An irreducible component E_i of E_Y is called a *rupture component* if it is not a rational curve or if it intersects at least three other components of the total transform. By definition a *curvetta* c_i of an irreducible component E_i of E_Y is a smooth curve germ that intersects transversally E_i at a smooth point of the total transform. To each irreducible component E_i of E we associate the rational number:

$$q_{E_i} = \frac{V_{f \circ \pi}(c_i)}{V_{g \circ \pi}(c_i)},$$

where $V_{f \circ \pi}(c_i)$ (resp. $V_{g \circ \pi}(c_i)$) is the order of $f \circ \pi$ (resp. $g \circ \pi$) on c_i . This quotient is called the *Hironaka quotient* of (f, g) on E_i .

This set of rational numbers associated to a complex analytic normal surface germ has been first introduced in [6]. It is shown, in particular, that if (u, v) are local coordinates of $(\mathbb{C}^2, 0)$ and if π is the minimal good resolution of ϕ , then the subset of Hironaka quotients associated to the rupture components of E_Y are topological invariants of (ϕ, u, v) . Another proof of this result is given in [12]. These results generalized the topological invariance of the polar quotients established first by M. Merle in [11] and by D.T. Lê, F. Michel and C. Weber in [7].

Remark 1. *By definition $V_{f \circ \pi}(c_i)$ is the evaluation on f of the divisorial valuation associated to E_i . Usually the irreducible component E_i is fixed and the associated divisorial valuation is defined as a valuation on the algebra of function germs on (X, p) . It is not our point of view. Here we fix the pair of functions (f, g) and we study the variation of the quotients q_{E_i} on the set of the irreducible components of the exceptional divisor. By definition the polar curve Γ_ϕ ,*

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first introduced in [5] and [16], (also called jacobian curve in [12]) is the union of the irreducible components of the singular locus of ϕ which are not contained in $\{fg = 0\}$. As explained in the end of this introduction and as illustrated by the examples of section 8, our results specify the position of Γ_ϕ . To understand the behaviour of Γ_ϕ is one of our purposes.

In this paper we study the growth behaviour of the Hironaka quotients of $\phi = (f, g)$ in the dual graph of a good resolution $R : (X', E_{X'}) \rightarrow (X, p)$ of the pair $(X, \{fg = 0\})$, obtained (see Section 2) using Hirzebruch-Jung's method. We also consider $\rho : (\tilde{X}, E_{\tilde{X}}) \rightarrow (X, p)$, the minimal good resolution of (X, p) such that the total transform of $\{fg = 0\}$ (by ρ) is a normal crossings divisor. By definition ρ is the minimal resolution of ϕ . But, X' dominates \tilde{X} by $\beta : X' \rightarrow \tilde{X}$ which is a sequence of blowing-downs of some specific irreducible components of $E_{X'}$ (see [4], Thm 5.9 or [1] p. 86). Then, we obtain similar results, on the growth behaviour of the Hironaka quotients of (f, g) , for the minimal resolution of ϕ and we can generalize them to any good resolution of ϕ .

Let $\pi : (Y, E_Y) \rightarrow (X, p)$ be a good resolution of ϕ . The weighted dual graph associated to π , denoted $G(Y)$, is constructed as follows. To each irreducible component E_i of the exceptional divisor E_Y we associate a vertex (i) weighted by its Hironaka quotient q_{E_i} . When two irreducible components of E_Y intersect, we join their associated vertices by edges which number is equal to the number of intersection points. When k ($k \geq 0$) irreducible components of the strict transform of $\{fg = 0\}$ meet E_i , we add k edges to the vertex (i) . If an edge represents the intersection point of an irreducible component of the strict transform of f (resp. g) with E_i , we endow the edge with a going-out arrow (resp. a going-in arrow (it means a reverse arrow)). By convention the Hironaka quotient of a going-in arrow is 0 and the Hironaka quotient of a going-out arrow is infinite.

Moreover by construction the graph $G(Y)$ is partially oriented as follows.

Let (e_{ij}) be an edge which represents a point of the intersection $E_i \cap E_j$. When $q_{E_i} = q_{E_j}$ the edge (e_{ij}) is not oriented. When $q_{E_i} < q_{E_j}$ then (e_{ij}) is oriented from (i) to (j) and we say that the edge e_{ij} is positively oriented.

Definition 2. A maximal arc in $G(Y)$ is a subgraph which is homeomorphic to a segment and which satisfies the following conditions:

- (1) it begins with a going-in arrow and ends with a going-out arrow,
- (2) it is a sequence of positively oriented edges,
- (3) the orientation of the edges induces a compatible positive orientation on the whole segment.



FIGURE 1. The two possible shapes of a maximal arc in $G(Y)$.

In this paper we prove the following theorem :

Theorem 3. Let $\phi = (f, g) : (X, p) \rightarrow (\mathbb{C}^2, 0)$ be a finite analytic morphism which is defined on a complex analytic normal surface germ (X, p) by two complex analytic function germs f and g . Let $\pi : (Y, E_Y) \rightarrow (X, p)$ be a good resolution of ϕ .

There exists a unique subgraph $A(Y)$ of $G(Y)$ which is a union of maximal arcs such that the Hironaka quotients of ϕ on the vertices of a connected component of the closure of $G(Y) \setminus A(Y)$ are constant.

Moreover $G(Y) \setminus A(Y)$ doesn't contain any arrow.

Remark 4. *Theorem 3 means that:*

An edge of $G(Y)$ is oriented if and only if it is contained in a maximal arc.

A vertex (i) of $G(Y)$ is in $A(Y)$ if and only if there exists at least one going-in arrow or edge arriving at (i) and at least one going-out arrow or edge leaving (i) .

In section 4 we show that $r : (Z, E_Z) \rightarrow (\mathbb{C}^2, 0)$, the minimal resolution of the discriminant union the axes (denoted Δ^+), gives a unique well-defined maximal arc $S(Z)$ in $G(Z)$. The pull-back of this maximal arc in the graph $G(X')$ of the Hirzebruch-Jung's resolution of ϕ (defined in section 2) gives the union of the maximal arcs $A(X')$.

Let (X, p) be an irreducible complex analytic surface germ (in particular, p is not necessarily an isolated singular point) and let $\phi : (X, p) \rightarrow (\mathbb{C}^2, 0)$ be a finite analytic morphism defined on (X, p) . Theorem 3 will also be true for any good resolution π of ϕ . Indeed any good resolution factorizes by the normalization $\nu : (\bar{X}, \bar{p}) \rightarrow (X, p)$ (see [4] Theorem 3.14). So we apply Theorem 3 for the finite morphism $\phi \circ \nu$ and we obtain:

Theorem. (generalized) *Let $\phi : (X, p) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism defined on an irreducible complex analytic surface germ (X, p) . Let $\pi : (Y, E_Y) \rightarrow (X, p)$ be a good resolution of ϕ . There exists a unique subgraph $A(Y)$ of $G(Y)$ which is a union of maximal arcs such that the Hironaka quotients of ϕ on the vertices of a connected component of the closure of $G(Y) \setminus A(Y)$ are constant.*

Moreover $G(Y) \setminus A(Y)$ doesn't contain any arrow.

To express the interest of Theorem 3 we first give an example (Example 1) where $(X, p) = (\mathbb{C}^2, 0)$. Theorem 3 is applied to show the growth of the Hironaka quotients of $\phi = (f, g)$. This behaviour of Hironaka quotients can not be obtained using the previous results of [10], because $\{f = 0\}$ and $\{g = 0\}$ have many branches with high contact.

One motivation to study Hironaka quotients is their relations with the Puiseux expansion of the branches of the discriminant of ϕ . The first Puiseux exponents of the discriminant of ϕ are the Hironaka quotients of the rupture vertices of the minimal resolution of ϕ (see [7] and [9] when (X, p) is non-singular and [6] and [12] for the general case).

Our Theorem 3, together with results on the rupture zones defined in [12], allows us to specify the position of the strict transform $\tilde{\Gamma}_\phi$ of the polar curve Γ_ϕ in the minimal good resolution of ϕ . In fact, $\tilde{\Gamma}_\phi$ meets the exceptional divisor in the rupture zones defined in [12]. But, these rupture zones are exactly the connected components of $G(\tilde{X}) \setminus A(\tilde{X})$ union the rupture components of $E_{\tilde{X}}$ which meet only irreducible components represented in $A(\tilde{X})$. We invite the reader to look first at examples of Section 8 where we describe $\tilde{\Gamma}_\phi$.

Another aspect of our work is linked to the Lipschitz geometry of the singularity. The inner and the outer bilipschitz classification of normal surfaces is based on the behavior of the polar and the discriminant of a generic plane projection of the surface (see [2] for the inner metric and [14] for the outer metric). But, for a generic plane projection the maximal arcs are degenerated. Then, as proved in [3], it is possible to twist the generic morphism ϕ and obtain finite morphisms with the same critical locus but different discriminants and non trivial $A(\tilde{X})$. With such constructions, it is possible to specify the position of the strict transform of the polar curve even for generic projections. We perform explicitly this process in Example 3 (Section 8).

Theorem 3 will be proved using Theorem 26 (Section 6.2) which determines the behaviour of Hironaka quotients in the Hirzebruch-Jung resolution of ϕ described in Section 2 (Definition 10). To prove Theorem 26, we relate the Hironaka quotients with the first Puiseux exponents of plane curve germs as follows.

Let (c, p) be a germ of irreducible curve on (X, p) which is not a branch of $\{fg = 0\}$. Let $\pi : (Y, E_Y) \rightarrow (X, p)$ be a good resolution of ϕ . Let (c_Y, z) be an irreducible component of the

strict transform of (c, p) in (Y, E_Y) , in particular $z \in E_Y$. As explained in Section 3, the first Puiseux exponent q_c of the plane curve germ $(\phi(c), 0) \subset (\mathbb{C}^2, 0)$ has the following behaviour:

if z is a smooth point of the total transform E_Y^+ and if E_i is the irreducible component of E_Y which contains z , then q_c is equal to the Hironaka quotient q_{E_i} of E_i .

if $z \in E_i \cap E_j$ and $q_{E_i} = q_{E_j}$, then $q_{E_i} = q_c = q_{E_j}$.

if $z \in E_i \cap E_j$ and $q_{E_i} < q_{E_j}$, then $q_{E_i} < q_c < q_{E_j}$.

This allows us to describe, in Lemma 21 (in Section 5), the growth behaviour of the Hironaka quotients associated to the minimal resolution of a quasi-ordinary normal surface germ.

In Section 2 we define the Hirzebruch-Jung resolution of ϕ and we describe its topological properties used in Sections 4 to 7.

In Section 7 we show how Theorem 3 can be deduced from Theorem 26.

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2. THE HIRZEBRUCH-JUNG'S RESOLUTION OF (X, p) ASSOCIATED TO ϕ

Let $\phi = (f, g) : (X, p) \rightarrow (\mathbb{C}^2, 0)$ be a finite analytic morphism which is defined on a complex analytic normal surface germ (X, p) by two complex analytic function germs f and g . In $(\mathbb{C}^2, 0)$, we fix the local coordinate system (u, v) . So, in $\phi(X)$, we have $u = f$ and $v = g$. The discriminant curve of ϕ is the image by ϕ of the critical locus $C(\phi)$ of ϕ . We denote Δ the union of the irreducible components of the discriminant curve which are not included in $\{uv = 0\}$.

We denote by $r : (Z, E_Z) \rightarrow (\mathbb{C}^2, 0)$ the minimal embedded resolution of $\Delta^+ = \Delta \cup \{uv = 0\}$ and $G(Z)$ its dual graph constructed as described in the introduction where f is replaced by u , g by v and we add an edge ended by a star for each irreducible component of the strict transform of Δ . Moreover we weight the vertex associated to an irreducible component D_i of E_Z by its Hironaka quotient q_{D_i} defined as follows:

$$q_{D_i} = \frac{V_{u \circ r}(c_i)}{V_{v \circ r}(c_i)}$$

where c_i is a curvetta of D_i .

We construct as in [8] and [15] a Hirzebruch-Jung resolution of $\phi : (X, p) \rightarrow (\mathbb{C}^2, 0)$. Here, we begin with the minimal resolution r of Δ^+ . The pull-back of ϕ by r is a finite morphism $\phi_r : (Z', E_{Z'}) \rightarrow (Z, E_Z)$ which induces an isomorphism from $E_{Z'}$ to E_Z . We denote r_ϕ the pull-back of r by ϕ , $r_\phi : (Z', E_{Z'}) \rightarrow (X, p)$.

$$\begin{array}{ccc} (Z', E_{Z'}) & \xrightarrow{r_\phi} & (X, p) \\ \downarrow \phi_r & & \downarrow \phi \\ (Z, E_Z) & \xrightarrow{r} & (\mathbb{C}^2, 0) \end{array}$$

FIGURE 2. The first step to construct the Hirzebruch-Jung resolution of ϕ .

In general Z' is not normal. Let $n : (\bar{Z}, E_{\bar{Z}}) \rightarrow (Z', E_{Z'})$ be its normalization.

Remark 5.

- (1) By construction, the discriminant locus of $\phi_r \circ n$ is included in $E_Z^+ = r^{-1}(\Delta^+)$ which is the total transform of $\Delta^+ = \Delta \cup \{uv = 0\}$ in Z .

- (2) Let $E_{Z'}^0$ be the open set of the points of $E_{Z'}$ which are smooth points in the total transform $E_{Z'}^+ = \phi_r^{-1}(E_Z^+)$. If $z' \in E_{Z'}^0$, there exists a small neighbourhood U' of z' in Z' which is a μ -constant family of curves parametrized by $U' \cap E_{Z'}$. Of course U' can be chosen such that $U' \cap E_{Z'}$ is a smooth disc in $E_{Z'}^0$. Therefore $n^{-1}(U')$ is a finite disjoint union of smooth germs of surface.
- (3) The restriction of the map $\phi_r \circ n$ to $E_{\bar{Z}}$ induces a finite morphism from $E_{\bar{Z}}$ to E_Z which is a regular covering on $E_{\bar{Z}}^0 = n^{-1}(E_Z^0)$.

Following Definitions 4.2 and 4.3 of [15], we state:

Definition 6. A germ (W, z) of surface is quasi-ordinary if there exists a finite morphism $\Phi : (W, z) \rightarrow (\mathbb{C}^2, 0)$ such that the discriminant locus of Φ is contained in a curve with normal crossings.

Moreover, if (W, z) is a normal quasi-ordinary surface germ, the point z is called a Hirzebruch-Jung singularity of (W, z) .

Lemma 7. Let P be a double point of E_Z^+ and \bar{P} a point of $(\phi_r \circ n)^{-1}(P)$. Then \bar{P} is a double point of $E_{\bar{Z}}^+$. Moreover, if \bar{P} is not a smooth point of \bar{Z} then \bar{P} is a Hirzebruch-Jung singularity of \bar{Z} .

Proof. Let P be a double point of E_Z^+ and $U(P)$ be a regular neighbourhood of P in Z . As Z is a smooth surface, $\partial U(P)$ is a 3-dimensional sphere. Let us show that $(\phi_r \circ n)^{-1}(P)$ is a union of double points of $E_{\bar{Z}}^+$. If $\bar{P} \in (\phi_r \circ n)^{-1}(P)$, let U be the connected component of $(\phi_r \circ n)^{-1}(U(P))$ that contains \bar{P} . Let $\phi_r \circ n|_U$ be the restriction of $\phi_r \circ n$ to the boundary ∂U of U . So, $\phi_r \circ n|_U : \partial U \rightarrow \partial U(P)$ is a finite ramified covering with ramification locus in $\partial U(P) \cap E_Z^+$. As Z is smooth and P is a double point of E_Z , then $\partial U(P) \cap E_Z^+$ is a Hopf link in the 3-sphere $\partial U(P)$. Therefore ∂U is a lens space that contains the link $\partial U \cap E_{\bar{Z}}^+$ included in two distinct irreducible components of $E_{\bar{Z}}^+$. Hence, \bar{P} is a double point of $E_{\bar{Z}}^+$. As the ramification locus of $\phi_r \circ n|_U : U \rightarrow U(P)$ is included in a normal crossing divisor, if \bar{P} is not a smooth point of \bar{Z} it is a Hirzebruch-Jung singularity.

As explained in Lemma 7, if \bar{z} is a singular point of \bar{Z} , then $(\phi_r \circ n)(\bar{z})$ is a double point of E_Z^+ . In particular, there are finitely many isolated singular points in \bar{Z} . The singularities of \bar{Z} are Hirzebruch-Jung singularities. More precisely, let $\bar{z}_i, 1 \leq i \leq n$, be the finite set of the singular points of \bar{Z} and (\bar{Z}_i, \bar{z}_i) a sufficiently small neighbourhood of \bar{z}_i in \bar{Z} . We have the following result (see [15] or [8] for a proof):

Theorem. The exceptional divisor of the minimal resolution of (\bar{Z}_i, \bar{z}_i) is a normal crossings divisor, each irreducible component of its exceptional divisor is a smooth rational curve, and its resolution dual graph is a bamboo (it means is homeomorphic to a segment).

Remark 8. In \bar{Z} an irreducible component of the strict transform of $\{fg = 0\}$ is not necessarily a curvetta of an irreducible component of the exceptional divisor. But the normalization morphism n has separated the irreducible components of the strict transform of $\{f = 0\}$ from those of $\{g = 0\}$.

Let $\bar{\rho}_i : (Z_i'', E_{Z_i}'') \rightarrow (\bar{Z}_i, \bar{z}_i)$ be the minimal resolution of the singularity (\bar{Z}_i, \bar{z}_i) . From [8] (Corollary 1.4.3), see also [15] (paragraph 4), the spaces Z_i'' and the maps $\bar{\rho}_i$ can be glued for $1 \leq i \leq n$, in a suitable way to give a smooth space X' and a map $\bar{\rho} : (X', E_{X'}) \rightarrow (\bar{Z}, E_{\bar{Z}})$ satisfying the following property :

Proposition 9. The map $r_\phi \circ n \circ \bar{\rho} : (X', E_{X'}) \rightarrow (X, p)$ is a good resolution of the singularity (X, p) in which the strict transform of $\{fg = 0\}$ is a normal crossings divisor.

Proof. The resolution r separates the strict transform of $\{u = 0\}$ from the one of $\{v = 0\}$. All the branches of the strict transform of $\{g = 0\}$ (resp. $\{f = 0\}$) by r_ϕ meet $E_{Z'}$ at the same point P' (resp. Q') and $P' \neq Q'$ because $\phi_r(P') \neq \phi_r(Q')$. The normalization morphism n separates the irreducible components of the strict transform of $\{f = 0\}$ (resp. of $\{g = 0\}$). In \bar{Z} , let \bar{P} be the intersection point of an irreducible component of the strict transform of $\{f = 0\}$ (resp. $\{g = 0\}$) with $E_{\bar{Z}}$. Let $P = (\phi_r \circ n)(\bar{P})$, $U(P)$ a regular neighbourhood of P in Z and U the connected component of $(\phi_r \circ n)^{-1}(U(P))$ that contains \bar{P} . $U(P)$ is a smooth surface germ that contains the double point P . Then from lemma 7, \bar{P} is either a smooth point of \bar{Z} , either a Hirzebruch-Jung singularity of \bar{Z} . In the second case, $\bar{\rho}$ is a resolution of \bar{P} .

Let us denote $R := r_\phi \circ n \circ \bar{\rho}$.

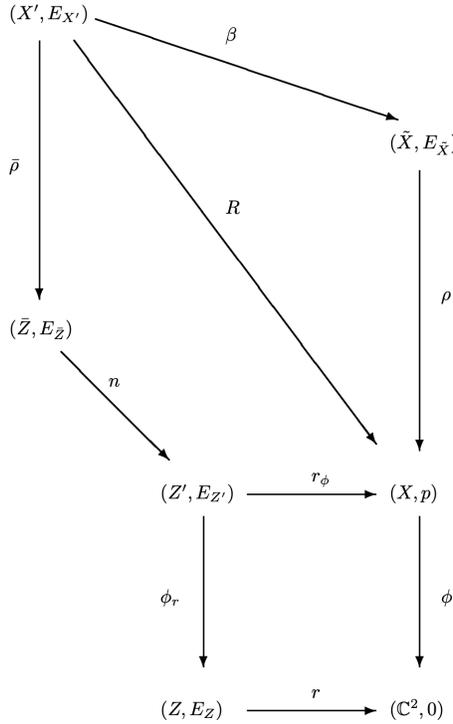


FIGURE 3. The commutative diagram of the morphisms involved in the Hirzebruch-Jung resolution of ϕ .

As R is the composition of three well defined morphisms, we can use the following definition which is a relative (to $\{fg = 0\}$) version of the classical Hirzebruch-Jung resolution of a normal germ of surface. By Proposition 9, R is a good resolution of ϕ .

Definition 10. *The morphism $R : (X', E_{X'}) \rightarrow (X, p)$ is the Hirzebruch-Jung resolution associated to ϕ .*

Now we can use the following result (for a proof see [4], Theorem 5.9, p.87):

Theorem. *Let $\rho : (\tilde{X}, E_{\tilde{X}}) \rightarrow (X, p)$ be the minimal good resolution of ϕ . There exists $\beta : (X', E_{X'}) \rightarrow (\tilde{X}, E_{\tilde{X}})$ such that $\rho \circ \beta = R$ and the map β consists in a composition of*

blowing-downs of irreducible components, of the successively obtained exceptional divisors, of self-intersection -1 , genus 0 and which are not rupture components.

3. THE QUOTIENTS ASSOCIATED TO THE MORPHISM ϕ

Let (c, p) be a germ of irreducible curve on (X, p) which is not a branch of $\{fg = 0\}$. Let $V_f(c)$ (resp. $V_g(c)$) be the order of f (resp. g) on c .

Definition 11. *The contact quotient q_c of (c, p) associated to the morphism $\phi = (f, g)$ is equal to:*

$$q_c = \frac{V_f(c)}{V_g(c)}.$$

The following remark relates the contact quotient of a germ (c, p) in (X, p) with the first Puiseux exponent of the direct image of (c, p) by ϕ .

Remark 12. *For local coordinates (u, v) of $(\mathbb{C}^2, 0)$ such that $u \circ \phi = f$ and $v \circ \phi = g$, let $u = a_0v^{m/n} + a_1v^{(m+1)/n} + \dots$, $a_0 \neq 0$, be a Puiseux expansion of $\phi(c)$. The definition of q_c implies that q_c is equal to the first Puiseux exponent of $\phi(c)$:*

$$q_c = \frac{m}{n} = \frac{V_u(\phi(c))}{V_v(\phi(c))}.$$

When the strict transform of a germ (c, p) in a good resolution $\pi : (Y, E_Y) \rightarrow (X, p)$ of (X, p) meets E_Y^+ at a smooth point, then q_c is an Hironaka quotient. More precisely, in [12] (Proposition 2.1) we have the following result:

Proposition 13. *Let $\pi : (Y, E_Y) \rightarrow (X, p)$ be a good resolution of ϕ and E_i be an irreducible component of E_Y of Hironaka quotient q_{E_i} . We denote by E_i^0 the smooth points of E_i in the total transform by π of $\{fg = 0\}$. Let x be a point of E_i^0 or a point of $E_i \cap E_j$ where $q_{E_j} = q_{E_i}$ and let (ξ, x) be an irreducible germ of curve at x . Then the contact quotient $q_{\pi(\xi)}$ of $(\pi(\xi), p)$ is equal to the Hironaka quotient on E_i i.e. :*

$$q_{\pi(\xi)} = q_{E_i}.$$

Moreover, if $x \in E_i \cap E_j$ where $q_{E_j} < q_{E_i}$, then $q_{E_i} < q_{\pi(\xi)} < q_{E_j}$.

Let $r : (Z, E_Z) \rightarrow (\mathbb{C}^2, 0)$ be the minimal embedded resolution of $\Delta^+ = \Delta \cup \{uv = 0\}$. Let D_i be an irreducible component of E_Z and c_i a curvetta of D_i .

Definition 14. *The Hironaka quotient of D_i , denoted q_{D_i} , is equal to:*

$$q_{D_i} = \frac{V_{u \circ r}(c_i)}{V_{v \circ r}(c_i)}.$$

Remark 15. *Let $(\gamma, 0)$ be an irreducible curve germ in $(\mathbb{C}^2, 0)$ which admits*

$$u = a_0v^{m/n} + a_1v^{(m+1)/n} + \dots, \quad a_0 \neq 0,$$

as Puiseux expansion. Let (C, z) be the strict transform by r of $(\gamma, 0)$ in (Z, E_Z) . Then

$$\frac{m}{n} = \frac{V_{u \circ r}(C)}{V_{v \circ r}(C)}.$$

Hence, if z is a smooth point of an irreducible component D_i of E_Z , we have $\frac{m}{n} = q_{D_i}$.

The following lemma is quite obvious, but very useful for computation of Hironaka quotients.

Lemma 16. *Let (c', p') be a germ of curve at p' . Let $\alpha : (c', p') \rightarrow (X, p)$ be a holomorphic germ which is a ramified covering over (c, p) of generic degree k and ramification locus p' . We have:*

$$q_c = \frac{V_f(c)}{V_g(c)} = \frac{V_{f \circ \alpha}(c')}{V_{g \circ \alpha}(c')}.$$

Proof. We have the following orders of functions:

$$V_{f \circ \alpha}(c') = k(V_f(c)) \text{ and } V_{g \circ \alpha}(c') = k(V_g(c)).$$

As in Section 2, we denote by $\rho : (\tilde{X}, E_{\tilde{X}}) \rightarrow (X, p)$ the minimal good resolution of ϕ and by $R : (X', E_{X'}) \rightarrow (X, p)$ the Hirzebruch-Jung resolution of ϕ .

Using the above Remark 12 and Lemma 16, and Proposition 13, we obtain the following behaviour of the Hironaka quotients for the divisors and morphisms involved in the diagram of Figure 3.

Lemma 17. *Let E'_i be an irreducible component of $E_{X'}^+$.*

If $(\phi_r \circ n \circ \bar{\rho})(E'_i)$ is an irreducible component D_i in E_Z^+ , then $q_{E'_i} = q_{D_i}$.

If $(\phi_r \circ n \circ \bar{\rho})(E'_i) \in D_i \cap D_j$ with $q_{D_i} = q_{D_j}$, then $q_{E'_i} = q_{D_i} = q_{D_j}$.

If $(\phi_r \circ n \circ \bar{\rho})(E'_i) \in D_i \cap D_j$ with $q_{D_i} < q_{D_j}$, then $q_{D_i} < q_{E'_i} < q_{D_j}$.

When $\beta(E'_i)$ is an irreducible component \tilde{E}_i of $E_{\tilde{X}}$, we have $q_{E'_i} = q_{\tilde{E}_i}$.

4. THE MAXIMAL ARC FOR THE MINIMAL RESOLUTION OF Δ^+

As in Section 2, $r : (Z, E_Z) \rightarrow (\mathbb{C}^2, 0)$ is the minimal embedded resolution of

$$\Delta^+ = \Delta \cup \{uv = 0\}.$$

Let $G(Z)$ be its dual graph constructed as described in the introduction where f is replaced by u, g by v . We add an edge ended by a star for each irreducible component of the strict transform of Δ . Moreover we weight the vertex associated to an irreducible component D_i of E_Z by its Hironaka quotient q_{D_i} (see definition 14).

From Remark 12, the quotient $q_{D_i} = \frac{V_{u \circ r}(c_i)}{V_{v \circ r}(c_i)}$ is equal to the first Puiseux exponent of $(r(c_i), 0)$.

As Δ^+ is a plane curve germ, $G(Z)$ is a tree. We consider the subgraph $S(Z)$ of $G(Z)$ which is the geodesic beginning with the (reverse) arrow associated to v and ending at the arrow associated to u . We orient this geodesic from v to u .

Proposition 18. *The graph $G(Z)$ admits an unique maximal arc which is equal to $S(Z)$.*

Proof. As $G(Z)$ is a tree, $S(Z)$ is homeomorphic to a segment. Notice that $G(Z)$ has only two arrows (one associated to the strict transform of $\{u = 0\}$ and the other to the strict transform of $\{v = 0\}$), both of them contained in $S(Z)$. So $\overline{G(Z)} \setminus S(Z)$ doesn't contain any arrow.

If $\Delta = \Delta^+ = \{uv = 0\}$, $S(Z)$ has a unique vertex with $q_{D_1} = 1$.

Otherwise, we number the irreducible components of E_Z corresponding to the vertices of $S(Z)$ from v to u . Let (i) and $(i+1)$ be two consecutive vertices on $S(Z)$ which represent respectively the irreducible components D_i and D_{i+1} . We have to prove that $q_{D_i} < q_{D_{i+1}}$.

Let c_i (resp. c_{i+1}) be a curvetta of D_i (resp. D_{i+1}). There exists $a_{i,0} \neq 0$ (resp. $a_{i+1,0} \neq 0$) such that the curve $r(c_i)$ (resp. $r(c_{i+1})$) admits the following Puiseux expansion:

$$u = a_{i,0}v^{m_i/n_i} + a_{i,1}v^{(m_i+1)/n_i} + \dots \quad (\text{resp. } u = a_{i+1,0}v^{m_{i+1}/n_{i+1}} + a_{i+1,1}v^{(m_{i+1}+1)/n_{i+1}} + \dots)$$

Statement*: The resolution of plane curve singularities computed by continued fraction expansion implies that $m_i/n_i < m_{i+1}/n_{i+1}$ (for example see [13], ch. 6). Indeed:

Let r_1 be the blowing-up of the origin in \mathbb{C}^2 . When $1 \leq m_i/n_i$, in local coordinates we write $r_1(x, y) = (xy, y)$. Then the first Puiseux exponent of the strict transform by r_1 of $r(c_i)$ is equal to $(m_i - n_i)/n_i$. When $m_i/n_i < 1$, we inverse the coordinates for the Puiseux expansion of $r(c_i)$. The first Puiseux exponent of $r(c_i)$ becomes $1 \leq n_i/m_i$. In local coordinates we write $r_1(x, y) = (x, xy)$, then the first Puiseux exponent of the strict transform by r_1 of $r(c_i)$ is equal to $(n_i - m_i)/m_i$. As r is a sequence of blowing-ups of points, the above computations imply by induction that:

Let I be the smallest interval of the real line which contains all the slow approximation values of the quotients m_i/n_i in their continued fraction expansions. Let $S(Z)^*$ be the graph $S(Z)$ minus the two arrows associated to $\{uv = 0\}$. There exists a homeomorphism $\iota : I \rightarrow S(Z)^*$ such that $\iota^{-1}((i)) = m_i/n_i$.

But $m_i/n_i = q_{D_i}$ and $m_{i+1}/n_{i+1} = q_{D_{i+1}}$. So, $S(Z)$ is a maximal arc as defined in the introduction.

It leaves to show that on a connected part T of $\overline{G(Z)} \setminus \overline{S(Z)}$ the Hironaka quotients are constant.

The intersection of T with $S(Z)$ is composed of an unique vertex. Let us call it (i) . An irreducible component D_j associated to a vertex of T is obtained by a sequence of blowing-ups of points which begins with the blowing-up of a smooth point (in the total transform of $\{uv = 0\}$) of D_i . Using Proposition 13, we obtain $q_{D_j} = q_{D_i}$.

Before describing the behaviour of the Hironaka quotients associated to $(X', E_{X'})$, we need to study the quotients associated to the minimal good resolution of a quasi-ordinary normal surface germ. We will use it to compute the Hironaka quotients on the irreducible components of $E_{X'}$ created by $\bar{\rho}$.

5. QUOTIENTS ASSOCIATED TO THE MINIMAL RESOLUTION OF A QUASI-ORDINARY NORMAL SURFACE GERM

Let $\Phi : (W, z) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism defined on a quasi-ordinary normal surface germ such that the discriminant locus is the union of the two coordinate axes of \mathbb{C}^2 .

Let us denote (u, v) the coordinate of \mathbb{C}^2 .

Remark 19. *The link of W is connected because (W, z) is an irreducible germ of complex surface. As $\{uv = 0\}$ is the discriminant locus of $\Phi : (W, z) \rightarrow (\mathbb{C}^2, 0)$, the topology of the situation implies that $\Phi^{-1}(\{u = 0\})$ (resp. $\Phi^{-1}(\{v = 0\})$) is an irreducible germ of curve in (W, z) .*

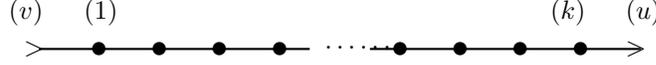
Proposition 20. *(see Theorem 1.4.2 of [8]) (W, z) has a minimal good resolution $\rho_W : (\tilde{W}, E_{\tilde{W}}) \rightarrow (W, z)$ such that:*

I) the dual of $E_{\tilde{W}}$ is a bamboo and all the vertices represent a rational smooth curve.

Let k be the number of irreducible components of $E_{\tilde{W}}$. We orient the bamboo from the vertex (1) to the vertex (k) . The vertices are indexed by this orientation.

II) the strict transform of $\Phi^{-1}(\{v = 0\})$ (resp. $\Phi^{-1}(\{u = 0\})$) is a curvetta of the irreducible component $E_1^{\tilde{W}}$ (resp. $E_k^{\tilde{W}}$) of $E_{\tilde{W}}$.

To obtain the dual graph $G(\tilde{W})$ we add to the vertex (1) (resp. (k)) of the dual graph of $E_{\tilde{W}}$ a reverse arrow indices by (v) which represents the strict transform (by $\Phi \circ \rho_{\tilde{W}}$) of $\{v = 0\}$ (resp. an arrow indices by (u) which represents the strict transform of $\{u = 0\}$). We get a graph which has the following shape:



The total transform of $\{uv = 0\}$ is $E_{\tilde{W}}^+ = (\Phi \circ \rho_W)^{-1}(\{uv = 0\})$. For all i , $1 \leq i \leq k$, let x_i be a point of $E_i^{\tilde{W}}$ which is smooth in $E_{\tilde{W}}^+$ and (c_i, x_i) be a curvetta of $E_i^{\tilde{W}}$. Let Γ be the union of the plane curve germs $\gamma_i = (\Phi \circ \rho_W)(c_i)$ and $\Gamma^+ = \Gamma \cup \{uv = 0\}$.

Lemma 21. *Let $\frac{m_i}{n_i}$ be the first Puiseux exponent of γ_i . For all i , $1 \leq i < k$, we have $\frac{m_i}{n_i} < \frac{m_{i+1}}{n_{i+1}}$.*

Remark 22. *From Section 3, the rational number $\frac{m_i}{n_i} \in \mathbb{Q}_+$ is the Hironaka quotient $q_{E_i^{\tilde{W}}}$ of Φ on $E_i^{\tilde{W}}$.*

Proof of Lemma 21. We order the set $Q = \{\frac{m_i}{n_i}\}$ of the first Puiseux exponent of the irreducible components of Γ . We obtain $Q = \{s_1 < \dots < s_j < \dots < s_{k'}\}$ where $k' \leq k$, $s_j \in \mathbb{Q}_+$. So, for all j , $1 \leq j \leq k'$, there exists at least one index $i(j)$ such that $\frac{m_{i(j)}}{n_{i(j)}} = s_j$.

The curve $\gamma_{i(j)}$ admits a Puiseux expansion which begins by:

$$u = a_{i(j)0} v^{m_{i(j)}/n_{i(j)}} + a_{i(j)1} v^{(m_{i(j)}+1)/n_{i(j)}} + \dots, a_{i(j)0} \neq 0$$

We will say that the plane curve germ $\gamma'_{i(j)}$, having

$$u = a_{i(j)0} v^{m_{i(j)}/n_{i(j)}}$$

as Puiseux expansion, is the shadow of $\gamma_{i(j)}$. Let Γ' be the union of the curves $\gamma'_{i(j)}$, $1 \leq j \leq k'$, and let r' be the minimal resolution of the plane curve germ $\Gamma'^+ = \Gamma' \cup \{uv = 0\}$:

$$r' : (M, E_M) \rightarrow (\mathbb{C}^2, 0).$$

The discriminant locus of Φ is $\{uv = 0\}$. Here, we begin with the resolution r' followed by a Hirzebruch-Jung construction to obtain a good resolution of (W, z) . It is described in Figure 4.

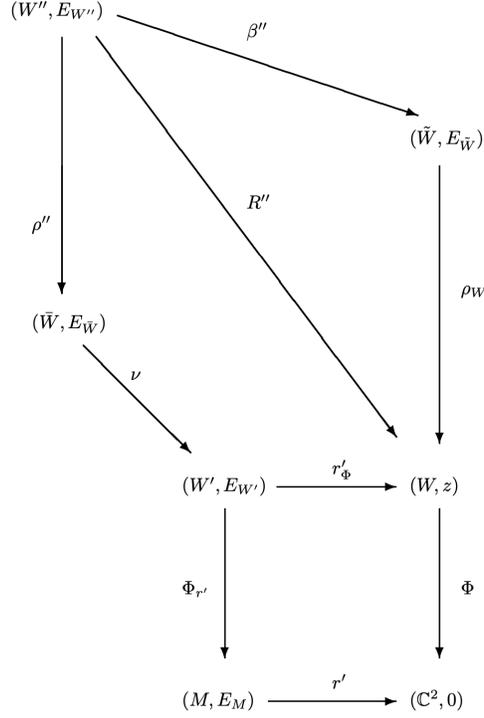


FIGURE 4. Diagram of the Hirzebruch-Jung’s resolution of (W, z) constructed with the curve $\Gamma^{'+} = \Gamma' \cup \{uv = 0\}$

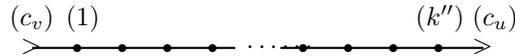
In Figure 4, $(W', E'_{W'})$ is the pull-back of Φ by r' . As explained in Section 2, the normalization $\nu : (\bar{W}, E_{\bar{W}}) \rightarrow (W', E'_{W'})$ followed by the minimal resolution

$$\rho'' : (W'', E_{W''}) \rightarrow (\bar{W}, E_{\bar{W}})$$

of the isolated singular points of \bar{W} is a good resolution $R'' : (W'', E''_{W'}) \rightarrow (W, z)$ of (W, z) . There exist $\beta'' : (W'', E''_{W'}) \rightarrow (\tilde{W}, E_{\tilde{W}})$ a composition of contraction of some irreducible components of $E''_{W'}$ such that $R'' = \rho_W \circ \beta''$.

Step I) By the minimal resolution of $\Gamma^{'+} = \Gamma' \cup \{uv = 0\}$, the dual graph $G(M)$ (endowed with an arrow (resp. a reverse arrow) representing the strict transform c_u , of $\{u = 0\}$ (resp. c_v of $\{v = 0\}$) is a bamboo with k'' ($k' \leq k''$) vertices. We orient this bamboo from the reverse arrow associated to c_v to the arrow associated to c_u . The indices (l) , $1 \leq l \leq k''$, of the vertices increase with this orientation.

The dual graph of E_M with the strict transforms c_u of $\{u = 0\}$ and c_v of $\{v = 0\}$ has the following shape:



The theory of the resolution of plane curve germ obtained by computation of the continued fraction expansion of the first Puiseux exponents s_j , $1 \leq j \leq k'$, implies that the strict transform

of $\gamma_{i(j)}$ and $\gamma'_{i(j)}$, having the same first Puiseux term and s_j as first Puiseux exponent meet the irreducible component $D_{l(j)}$ at the same smooth point of $D_{l(j)}$ in $E_M^+ = r'^{-1}\{uv = 0\}$. Moreover, we have $l(1) < l(2) < \dots < l(j) < \dots < l(k')$.

Step II) As $G(\bar{W})$ is a bamboo and as $\Phi^{-1}(\{u = 0\})$ (resp. $\Phi^{-1}(\{v = 0\})$) is an irreducible germ of curve in (W, z) , $\Phi_{r'} \circ \nu$ induces an isomorphism of graph $\nu_G : G(\bar{W}) \rightarrow G(M)$. Indeed, the strict transform of $\{v = 0\}$ (resp. $\{u = 0\}$) being irreducible,

$$(\Phi_{r'} \circ \nu)^{-1}(D_1)$$

(resp. $(\Phi_{r'} \circ \nu)^{-1}(D_{k'})$) is only one irreducible component of $E_{\bar{W}}$. But there is no cycle in the graph $G(W'')$ because $G(\bar{W})$ has no cycle. Moreover, ρ'' is only a resolution of quasi-ordinary singular points. The graph $G(W'')$ is obtained from $G(\bar{W})$ by replacing some edges by bamboos and $G(\bar{W})$ has no cycle. So, all the $(\Phi_{r'} \circ \nu)^{-1}(D_l)$ are irreducible in $E_{\bar{W}}$ and two irreducible components of $E_{\bar{W}}$ has at most one common point. We can identify $G(\bar{W})$ with $G(M)$ after putting, via ν_G , the orientation and indices of $G(M)$ on $G(\bar{W})$.

Step III) By step II, $G(\bar{W})$ is, in particular, a bamboo. The graph $G(W'')$ is obtained from $G(\bar{W})$ by replacing some edges by bamboos, it produces a new bamboo which is just an extension of $G(\bar{W})$. We lift the orientation of $G(\bar{W})$ on $G(W'')$ and we order the indices of the vertices of $G(W'')$ with the help of this orientation.

For all j , $1 \leq j \leq k'$, let $D_{l(j)}^0$ be the set of the smooth points of $D_{l(j)}$ in E_M^+ . The above description of $G(W'')$ implies that it exists only one index $l''(j)$ such that

$$(\Phi_{r'} \circ \nu \circ \rho'')^{-1}(D_{l(j)}^0) = E_{l''(j)}''^0$$

and the strict transform of $\gamma_{i(j)}$ via $(r' \circ \Phi_{r'} \circ \nu \circ \rho'')$ meets $E_{l''(j)}''^0$. But, $\gamma_{i(j)}$ is the direct image (by $\Phi \circ \rho_W$) of the chosen curvetta $(c_{i(j)}, x_{i(j)})$ of $E_{i(j)}^{\bar{W}}$. The commutation in the diagram (Figure 4) implies that $\beta''(E_{l''(j)}''^0) = E_{i(j)}^{\bar{W}}$. But, $(c_{i(j)}, x_{i(j)})$ is the only chosen curvetta of $E_{i(j)}^{\bar{W}}$. So, there is a bijection between the indices j and i . This implies: $k' = k$, $j = i(j) = i$, and for all i , $1 \leq i \leq k$,

$$s_i = \frac{m_i}{n_i} < s_{i+1} = \frac{m_{i+1}}{n_{i+1}}$$

This ends the proof of Lemma 21.

6. BEHAVIOUR OF THE HIRONAKA QUOTIENTS IN EACH STEP OF THE HIRZEBRUCH-JUNG RESOLUTION OF ϕ

Remark 23. *The computation of the Hironaka quotients of (f, g) in each step of the Hirzebruch-Jung resolution of ϕ is based on the following principle: Lemma 16 and Remark 12 (of Section 3), imply that the Hironaka quotient on an irreducible component E of the exceptional divisor $E_{\bar{Z}}$ (resp. $E_{X'}$) is equal to the contact quotient of the direct image, in (X, p) , of a curvetta of E .*

We will show how the Hironaka quotients associated to the irreducible components of E_Z enable us to describe the behaviour of the Hironaka quotients on the irreducible components of E_Z (resp. $E_{X'}$).

6.1. Hironaka quotients associated to \bar{Z} . Let \bar{E}_i be an irreducible component of $E_{\bar{Z}}$. Let \bar{E}_i^0 be the open set of the smooth points of \bar{E}_i in the total transform

$$E_{\bar{Z}}^+ = (r \circ \phi_r \circ n)^{-1}(\Delta^+).$$

By construction $(\phi_r \circ n)(\bar{E}_i^0)$ is the set D_i^0 of the smooth points, in the total transform $E_Z^+ = r^{-1}(\Delta^+)$, of an irreducible component D_i of E_Z .

Proposition 24. *The Hironaka quotient on \bar{E}_i is equal to the Hironaka quotient on D_i .*

Proof. Let $\bar{z} \in \bar{E}_i^0$ and let (\bar{c}_i, \bar{z}) be a curvetta of \bar{E}_i . In Section 3, we have seen that the Hironaka quotient $q_{\bar{E}_i}$ is equal to the first Puiseux exponent of $(\phi \circ r \circ \phi \circ n)(\bar{c}_i, \bar{z}) = (r \circ \phi_r \circ n)(\bar{c}_i, \bar{z})$. Let $z = (\phi_r \circ n)(\bar{z}) \in D_i^0$. But, as $(\gamma_i, z) = (\phi_r \circ n)(\bar{c}_i, \bar{z})$ is a germ of curve at a point of D_i^0 , this implies that $(r(\gamma_i), 0)$ has the same first Puiseux exponent than $(r(c_i), 0)$ where (c_i, z) is a curvetta (at z) of D_i and which is equal to the Hironaka quotient on q_{D_i} (see section 4). This proves that $q_{D_i} = q_{\bar{E}_i}$.

As described in the introduction, we construct the partially oriented dual graph $G(\bar{Z})$ of \bar{Z} . In Section 2 we have seen that: the pull-back of ϕ by r is a finite morphism

$$\phi_r : (Z', E_{Z'}) \rightarrow (Z, E_Z)$$

which induces an isomorphism from $E_{Z'}$ to E_Z . Moreover, Point 3 of Remark 5 states that the restriction of the map $\phi_r \circ n$ to $E_{\bar{Z}}$ induces a finite morphism from $E_{\bar{Z}}$ to E_Z which is a regular covering on $E_{\bar{Z}}^0 = n^{-1}(E_Z^0)$. The morphism of graphs $n_G : G(\bar{Z}) \rightarrow G(Z)$ induced by $\phi_r \circ n : (\bar{Z}, E_{\bar{Z}}) \rightarrow (Z, E_Z)$ is defined as follows: the vertices associated to the irreducible components of $(\phi_r \circ n)^{-1}(D_i)$ are sent to the vertex associated to D_i . An edge associated to a point of $(\phi_r \circ n)^{-1}(D_i \cap D_j)$ is sent to the edge associated to $D_i \cap D_j$. An edge of $G(\bar{Z})$ is oriented (with the same orientation) if and only if its image by n_G is oriented in $G(Z)$.

Proposition 24 can also be stated in terms of dual graphs as follows:

Remark 25. *The Hironaka quotient of the vertex $(i) \in G(\bar{Z})$ is equal to the Hironaka quotient of the vertex $n_G(i) \in G(Z)$. Hence, n_G is an orientation preserving morphism of graphs.*

6.2. Hironaka quotients associated to X' . As in Section 2, we consider the finite set $\bar{z}_i, 1 \leq i \leq n$, of the singular points of \bar{Z} . For each index i , we choose a sufficiently small neighbourhood (\bar{Z}_i, \bar{z}_i) of \bar{z}_i in \bar{Z} and we denote by $\bar{\rho}_i : (Z''_i, E_{Z''_i}) \rightarrow (\bar{Z}_i, \bar{z}_i)$ the minimal resolution of the singularity (\bar{Z}_i, \bar{z}_i) . We have seen (Section 2) that $\bar{z}_i = \bar{E}_{i_1} \cap \bar{E}_{i_2}$ is a double point of the total transform $E_{\bar{Z}}^+ = (\phi_r \circ n)^{-1}(E_{\bar{Z}}^+)$ (\bar{E}_{i_1} or \bar{E}_{i_2} may be an irreducible component of the strict transform of the discriminant Δ). As (\bar{Z}_i, \bar{z}_i) is a quasi-ordinary singular point, the dual graph of $E_{Z''_i} = (\bar{\rho})^{-1}(\bar{z}_i)$ is a bamboo, let us denote by k the number of its irreducible components.

Let us denote by E'_{i_1} (resp. E'_{i_2}) the irreducible component of $E_{X'}$ such that $(\bar{\rho})(E'_{i_1}) = \bar{E}_{i_1}$ (resp. $(\bar{\rho})(E'_{i_2}) = \bar{E}_{i_2}$). One extremity of this bamboo represents the irreducible component E'_1 of $E_{X'}$ which meets E'_{i_1} . We index it by (1). To obtain the dual graph $G(Z''_i)$, we add to (1) a reverse arrow indexed by (i_1) which represents E'_{i_1} .

The other extremity of this bamboo represents the irreducible component E'_k of $E_{X'}$ which meets E'_{i_2} . We index it by (k) . To obtain the dual graph $G(Z''_i)$, we add to (k) an arrow indexed by (i_2) which represents E'_{i_2} . We orient $G(Z''_i)$ from (1) to (k) and we order the indices of the vertices with the help of this orientation.

The graph $G(Z''_i)$ has the following shape:



Theorem 26. *Let E'_j be an irreducible component of $E_{X'}$ and let z' be a point of E_j^0 which is the set of the smooth points of E'_j in the total transform $(\bar{\rho})^{-1}(E_{\bar{Z}}^+)$. Let (c'_j, z') be a curvetta (at z') of E'_j .*

If $\bar{\rho}(z') \neq \bar{z}_i, \bar{z}_i \in \{\bar{z}_i, 1 \leq i \leq n\}$, we have the following equality of the Hironaka quotients: $q_{E'_j} = q_{\bar{E}_j} = q_{D_j}$.

If $\bar{\rho}(z') = \bar{z}_i$, we have seen that $\bar{z}_i = \bar{E}_{i_1} \cap \bar{E}_{i_2}$ is a double point of the total transform

$$E_Z^+ = (\phi_r \circ n)^{-1}(E_Z^+).$$

We have two cases

I) either $q_{\bar{E}_{i_1}} = q_{\bar{E}_{i_2}}$, then $q_{E'_j} = q_{\bar{E}_{i_1}} = q_{\bar{E}_{i_2}}$,

II) or $q_{\bar{E}_{i_1}} < q_{\bar{E}_{i_2}}$, then $q_{\bar{E}_{i_1}} < q_{E'_j} < q_{\bar{E}_{i_2}}$. More precisely, the dual graph of $(\bar{\rho})^{-1}(\bar{z}_i)$ is a bamboo. We orient this bamboo from the vertex (i_1) to the vertex (i_2) . With this orientation, we order the indices (j) of the dual graph of $(\bar{\rho})^{-1}(\bar{z}_i)$ from (1) to (k) . We have:

$$q_{E'_{i_1}} = q_{\bar{E}_{i_1}} < q_{E'_1} < \dots < q_{E'_j} < \dots < q_{E'_k} < q_{\bar{E}_{i_2}} = q_{E'_{i_2}}.$$

Proof of Theorem 26.

Let us recall that $q_{E'_j}$ is equal to the first Puiseux exponent of the plane curve germ γ_j where

$$\gamma_j = (r \circ \phi_r \circ n \circ \bar{\rho})(c'_j) = (\phi \circ r_\phi \circ n \circ \bar{\rho})(c'_j).$$

Suppose $\bar{\rho}(z') \neq \bar{z}_i, \bar{z}_i \in \{\bar{z}_i, 1 \leq i \leq n\}$, $\bar{\rho}(E'_j)$ is an irreducible component \bar{E}_j of E_Z and $(\phi_r \circ n \circ \bar{\rho})(E'_j) = D_j$ is an irreducible component of E_Z . Then, the first Puiseux exponent of $(r \circ \phi_r \circ n \circ \bar{\rho})(c'_j, z')$ is equal to $q_{\bar{E}_j} = q_{D_j}$.

If $\bar{\rho}(z') = \bar{z}_i$, we have seen that $\bar{z}_i = \bar{E}_{i_1} \cap \bar{E}_{i_2}$ is a double point of the total transform

$$E_Z^+ = (\phi_r \circ n)^{-1}(E_Z^+).$$

Let $D_{i_1} = (\phi_r \circ n)(\bar{E}_{i_1})$, $D_{i_2} = (\phi_r \circ n)(\bar{E}_{i_2})$ and $z = (\phi_r \circ n \circ \bar{\rho})(z') \in D_{i_1} \cap D_{i_2}$. In Section 6.1, Proposition 24, we proved that $q_{D_{i_1}} = q_{\bar{E}_{i_1}}$ and $q_{D_{i_2}} = q_{\bar{E}_{i_2}}$.

I) If $q_{\bar{E}_{i_1}} = q_{\bar{E}_{i_2}}$, we have $q_{D_{i_1}} = q_{D_{i_2}}$.

As $z = (\phi_r \circ n \circ \bar{\rho})(z') \in D_{i_1} \cap D_{i_2}$, we deduce from lemma 17 that $q_{E'_j} = q_{D_{i_1}} = q_{D_{i_2}}$.

II) If $q_{D_{i_1}} = q_{\bar{E}_{i_1}} < q_{\bar{E}_{i_2}} = q_{D_{i_2}}$, we have to study the minimal resolution

$$\bar{\rho}_i : (Z''_i, E_{Z''_i}) \rightarrow (\bar{Z}_i, \bar{z}_i)$$

of the singularity (\bar{Z}_i, \bar{z}_i) .

For each irreducible component E'_j of $(\bar{\rho})^{-1}(\bar{z}_i)$ we choose a curvetta (c'_j, z'_j) of E'_j . We take the following notation: $D_{i_1} = (\phi_r \circ n \circ \bar{\rho})(E'_{i_1})$ and $D_{i_2} = (\phi_r \circ n \circ \bar{\rho})(E'_{i_2})$. So, we have $z_i = (\phi_r \circ n \circ \bar{\rho})(z'_j) = D_{i_1} \cap D_{i_2}$.

But, $V_i = (\phi_r \circ n \circ \bar{\rho})(Z''_i)$ is a neighbourhood of z_i in Z . Let us denote by Φ_i the restriction of $(\phi_r \circ n \circ \bar{\rho})$ on Z''_i . As Z is smooth and E_Z is a normal crossing divisor in Z , the morphism Φ_i satisfies the hypothesis of Lemma 21 where the smooth plane curve germs (D_{i_1}, z_i) and (D_{i_2}, z_i) play the role of the two axes $u = 0$ and $v = 0$. With this choice of axes at z_i in V_i , the first Puiseux exponents s_j of $c_j = \Phi_i(c'_j)$ are strictly ordered $s_1 < \dots < s_j < \dots < s_k$.

The curve c_j admits a Puiseux expansion which begins by:

$$x = a_{j0}y^{m_j/n_j} + a_{j1}y^{(m_j+1)/n_j} + \dots, a_{j0} \neq 0, m_j/n_j = s_j.$$

We will say that the plane curve germ c_j^* , having

$$x = a_{j0}y^{m_j/n_j}$$

as Puiseux expansion, is the shadow of c_j . Let $\gamma_j^* = r(c_j^*)$ and let t_j be the first Puiseux exponent of γ_j^* .

As $q_{D_{i_1}} < q_{D_{i_2}}$, the edge which represents $z_i = D_{i_1} \cap D_{i_2}$ in $G(Z)$ is an edge of the maximal arc $S(Z)$ of Proposition 18, Section 4. As explained in Statement* in the proof of Proposition 18 (Section 4), the resolution of plane curve germs implies that $q_{D_{i_1}} < t_1 < \dots < t_j < \dots < t_k < q_{D_{i_2}}$. Moreover t_j is also the first Puiseux exponent of $r(c_j) = \gamma_j$. But q_{E_j} is equal to the first Puiseux exponent of γ_j . This ends the proof of Theorem 26.

7. BEHAVIOUR OF THE DUAL GRAPHS IN EACH STEP OF THE HIRZEBRUCH-JUNG RESOLUTION OF ϕ

7.1. Maximal arcs in the dual graph $G(\bar{Z})$ of the normalization. Let $A(\bar{Z}) := n_G^{-1}(S(Z))$ be the inverse image of the maximal arc $S(Z)$ of $G(Z)$.

Theorem 27. *$A(\bar{Z})$ is the union of all the maximal arcs of $G(\bar{Z})$. The Hironaka quotients of the vertices of a connected component of the closure of $G(\bar{Z}) \setminus A(\bar{Z})$ are constant.*

Moreover $G(\bar{Z}) \setminus A(\bar{Z})$ doesn't contain any arrow.

Proof. By remark 25, n_G preserves the Hironaka quotients. The definition of $A(\bar{Z})$ implies that the Hironaka quotients of the vertices of a connected component of the closure of $G(\bar{Z}) \setminus A(\bar{Z})$ are constant.

Again by remark 25, an edge of $G(\bar{Z})$ is oriented if and only if it is an edge of $A(\bar{Z})$. All the going-in arrows (resp. going-out) arrows are in $A(\bar{Z})$ because there are above (by n_G) the unique going-in (resp. going-out) arrow of $S(Z)$. The image, by n_G , of a vertex of $A(\bar{Z})$ is a vertex of $S(Z)$. In $S(Z)$ a vertex has exactly one going-in edge and one going-out edge. As n_G preserves the orientation of the edges, a vertex of $A(\bar{Z})$ has at least a going-in edge and a going-out edge. It allows us to show that each edge and each vertex of $A(\bar{Z})$ belong to a maximal arc of $A(\bar{Z})$.

7.2. Maximal arcs in the dual graph $G(X')$ of the minimal resolution of the Hirzebruch-Jung singularities of \bar{Z} . If an edge e of $G(\bar{Z})$ represents a Hirzebruch-Jung singular point \bar{z} of \bar{Z} , then in $G(X')$ this edge is replaced by an open bamboo. The morphism $\bar{\rho}_G : G(X') \rightarrow G(\bar{Z})$ induced by $\bar{\rho}$ is defined as follows: the open bamboo of $G(X')$ which represents the interior of the resolution graph of \bar{z} is sent to the edge e of $G(\bar{Z})$. The restriction of $\bar{\rho}_G$ on the complement of the union of the interior of these bamboos, created by $\bar{\rho}$, is a bijection on the complement, in $G(\bar{Z})$, of the union of all the edges associated to a Hirzebruch-Jung singularity of \bar{Z} .

Let $A(X') := (\bar{\rho}_G)^{-1}(A(\bar{Z}))$ be the inverse image of $A(\bar{Z})$ by the morphism of graphs $\bar{\rho}_G : G(X') \rightarrow G(\bar{Z})$ induced by $\bar{\rho}$.

Theorem 28. *$A(X')$ is the union of all the maximal arcs of $G(X')$. The Hironaka quotients of the vertices of a connected component of the closure of $G(X') \setminus A(X')$ are constant.*

Moreover $G(X') \setminus A(X')$ doesn't contain any arrow.

Proof. The graph $G(X')$ is obtained from $G(\bar{Z})$ as follows.

If an edge of $G(\bar{Z})$ represents a point which is a smooth point of \bar{Z} , then we keep this edge in $G(X')$ and its extremities has the same Hironaka quotients.

If an edge e of $G(\bar{Z})$ represents a Hirzebruch-Jung singular point of \bar{Z} , then in $G(X')$ this edge is replaced by a bamboo. If e is not oriented, from point I) of theorem 26, the Hironaka quotients are constant on the closure of the bamboo. So the closure of the bamboo is in the closure of $G(X') \setminus A(X')$. If e is oriented, from point II) of theorem 26, the bamboo has the same orientation and is included in $A(X')$ by construction. In particular, the inverse image by $\bar{\rho}_G$ of a maximal arc of $A(\bar{Z})$ is a maximal arc of $A(X')$.

7.3. Maximal arcs in the dual graph $G(Y)$ of a good resolution of ϕ . Let

$$\beta' : (X', E_{X'}) \rightarrow (X_1, E_{X_1})$$

be the contraction of an irreducible component of $E_{X'}$ of self-intersection -1 which is not a rupture component. A maximal sequence of such blowing-downs gives a morphism

$$\beta : (X', E_{X'}) \rightarrow (\tilde{X}, E_{\tilde{X}}).$$

Then the contraction of $E_{\tilde{X}}$ denoted $\rho : (\tilde{X}, E_{\tilde{X}}) \rightarrow (X, p)$ is the minimal resolution of ϕ (see [4] (Theorem 5.9 p. 87) or [1] (Theorem 6.2 p. 86)).

Lemma 29. *Let $\phi = (f, g) : (X, p) \rightarrow (\mathbb{C}^2, 0)$ be a finite analytic morphism which is defined on a complex analytic normal surface germ (X, p) by two complex analytic function germs f and g . Let $\pi_i : (Y_i, E_{Y_i}) \rightarrow (X, p)$, $i = 1, 2$, be two good resolutions of ϕ . If there exists a morphism $\beta : (Y_2, E_{Y_2}) \rightarrow (Y_1, E_{Y_1})$ which is the contraction of an irreducible component E of E_{Y_2} of self-intersection -1 and such that E is not a rupture component, then:*

The graph $G(Y_1)$ satisfies Theorem 3 if and only if $G(Y_2)$ satisfies Theorem 3.

Remark 30. *By Theorem 28, the Hirzebruch-Jung resolution of ϕ has a graph which satisfies Theorem 3. By finite iterations, the above lemma gives a proof of Theorem 3 for the graph of the minimal resolution of ϕ .*

Let $\pi : (Y, E_Y) \rightarrow (X, p)$, be any good resolution of ϕ . As proved in [4] (Theorem 5.9, p.87) and [1] (Theorem 6.2 p. 86), there exists a sequence of contractions γ of irreducible components of self-intersection -1 which are not rupture components such that the following diagram commutes.

$$\begin{array}{ccc} (Y, E_Y) & \xrightarrow{\gamma} & (\tilde{X}, E_{\tilde{X}}) \\ \pi \searrow & & \swarrow \rho \\ & (X, p) & \end{array}$$

If Theorem 3, is proved for $G(\tilde{X})$, we iterate Lemma 29 to prove Theorem 3 for $G(Y)$.

Proof of Lemma 29. Let $\beta_G : G(Y_2) \rightarrow G(Y_1)$ be the morphism of graphs induced by β .

First, we suppose that $G(Y_1)$ satisfies Theorem 3. But, β is the blowing up of a point z of the exceptional divisor E_{Y_1} .

If z is a smooth point, in the total transform $E_{Y_1}^+ \cap \{fg = 0\}$, of an irreducible component E' of E_{Y_1} , we obtain E_{Y_2} from E_{Y_1} by adding the new irreducible component E which only meet E' in one smooth point. As stated in Proposition 13, E and E' have the same Hironaka quotient. Then, $G(Y_2)$ is obtained from $G(Y_1)$ by adding a small bamboo B consisting in one new vertex and one new non-oriented edge which represents the intersection $E \cap E'$. So, $A(Y_2) = (\beta_G^{-1}(A(Y_1)) \setminus B)$ satisfies Theorem 3.

If z is a double point of $E_{Y_1}^+$, let us denote e the edge of $G(Y_1)$ which represents z . In this case, will prove that $A(Y_2) = (\beta_G^{-1}(A(Y_1)))$ is the union of the maximal arcs of $G(Y_2)$. If $e \in G(Y_1) \setminus A(Y_1)$, then $z \in E' \cap E''$ where E' and E'' are two irreducible component of E_{Y_1} which have the same Hironaka quotient q . By Proposition 13, q is also the Hironaka quotient of E . Then, $\beta_G^{-1}(e) \subset \beta_G^{-1}(G(Y_1) \setminus A(Y_1))$ consists in the vertex associated to E and two non-oriented edges. It proves than $\beta_G^{-1}(e)$ are not included in a maximal arc. If $e \in A(Y_1)$, e is oriented. Then, we have $q' < q''$ where q' is the Hironaka quotient of E' (resp. q'' of E'') and e represents $E' \cap E''$. By Proposition 13, $q' < q < q''$. The argument is the same when E' (resp. E'') are

an irreducible component of the strict transformed of $\{fg = 0\}$. In Figure 5, it is shown how β_G^{-1} extends a maximal arc containing e .

Conversely, we suppose that there exists in $G(Y_2)$ a union of maximal arcs $A(Y_2)$ which satisfies Theorem 3. Let us denote v the vertex of $G(Y_2)$ which represents the irreducible component E contracted by β . We prove that $A(Y_1) = \beta_G(A(Y_2))$ is the union of maximal arcs which satisfies Theorem 3 as follows:

If E meets only one irreducible component of $E_{Y_2}^+$, this neighbour is not an arrow because $G(Y_2)$ is a connected graph. Only one edge arrives to v . Remark 4 implies that this only edge which arrives to the vertex v is not in a maximal arc. So, $G(Y_1)$ is obtained from $G(Y_2)$ by deleting the vertex v and the only non-oriented edge which meets v . The morphism β_G restricted on $A(Y_2)$ induces an isomorphism to $A(Y_1)$.

If E meets two irreducible components of $E_{Y_2}^+$, then two cases occur.

a) If v is not a vertex of $A(Y_2)$, then from Remark 4, the two edges which meet v are not oriented and β_G deletes v and the two edges are replaced by a unique non oriented edge. In this case, The morphism β_G restricted on $A(Y_2)$ induces an isomorphism to $A(Y_1)$.

b) If v is a vertex of $A(Y_2)$, notice first that the restriction of β_G on $G(Y_2) \setminus A(Y_2)$ induces an isomorphism to $G(Y_1) \setminus A(Y_1)$. On the other hand, v meets at most one arrow because E is not a rupture component and E meets at most one other irreducible component of E_{Y_2} . The action of β_G on a maximal arc which contains v deletes v and the contraction of $A(Y_2)$ to $A(Y_1)$ around v has one of the three following shapes:

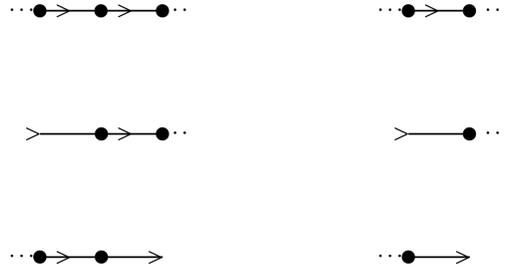


FIGURE 5. The shapes of the possible contractions of $A(Y_2)$ in $A(Y_1)$

Obviously β_G contracts a maximal arc containing v into a maximal arc. This ends the proof of Lemma 29.

8. EXAMPLES

Convention for the figures of this section: the irreducible components of the strict transform of Δ , in the minimal resolution of Δ^+ , are represented by edges ended with a star.

8.1. **Example 1.** Let $\phi = (f, g) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be defined by

$$f(x, y) = (x^2 - y^3)y(y + x^5)(x + y + x^3) \text{ and } g(x, y) = x(y + 2x^5)(x + y).$$

The critical locus of ϕ admits five irreducible components. Four of them are smooth and tangent to $\{y = -x\}$, $\{y = x\}$ and $\{y = 0\}$. The fifth one is tangent to $\{x = 0\}$ and topologically equivalent to $\{x^2 - y^3 = 0\}$.

The graph $G(Z)$ is in Figure 6.

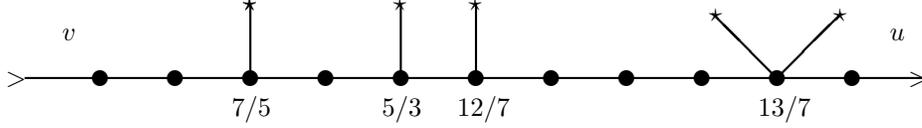


FIGURE 6. Graph of the minimal resolution r of Δ^+

Each vertex of $G(Z)$ belongs to the maximal arc $S(Z)$.

The graph of the minimal resolution ρ , weighted with the Hironaka quotients of (f, g) , is in Figure 7.

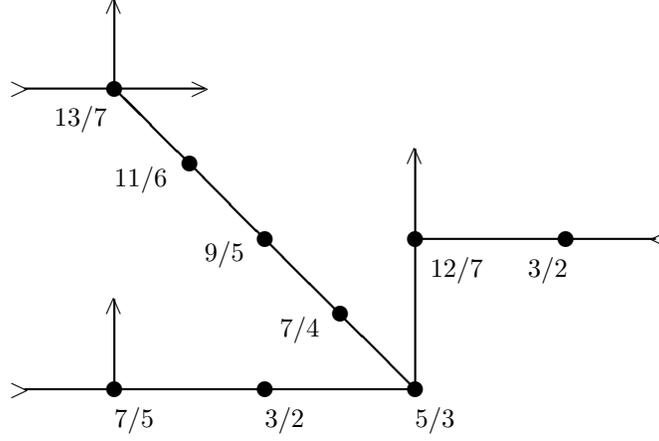


FIGURE 7. Graph of the minimal resolution of ϕ

In this example, the subgraph $A(\tilde{X})$ coincides with $G(\tilde{X})$.

So the rupture zones are reduced to the rupture vertices. Here we have for rupture vertices the ones of quotients $\{7/5, 5/3, 12/7, 13/7\}$. Considering only Figure 7, we can say that the strict transform $\tilde{\Gamma}_\phi$ meets $E_{\tilde{X}}$ at a smooth point of the four irreducible rupture components.

Notice that the subgraphs $A(\tilde{X})$ and $G(\tilde{X})$ will have similar shapes when

$$f(x, y) = (x^2 - y^3)y(y + x^k)(x + y + x^l) \text{ and } g(x, y) = x(y + 2x^k)(x + y)$$

where k, l are integers strictly greater than 1.

8.2. Example 2. Let us consider the surface $(X, 0)$ of equation:

$$z^3 = (y^3 - x^2)(y^3 - (x + y)^2)$$

and let $\phi : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be the projection on the (x, y) -plane. Notice that this projection is not a generic one.

The discriminant locus of ϕ is $\Delta : (v^3 - u^2)(v^3 - (u + v)^2) = 0$. The minimal resolution tree of Δ^+ is in Figure 8.

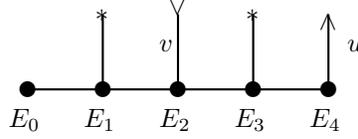


FIGURE 8. Graph of the minimal resolution r of Δ^+

The vertices E_2, E_3, E_4 belong to the maximal arc $S(Z)$.

The set of Hironaka quotients associated to $G(Z)$ is $\left\{1, \frac{3}{2}, 2\right\}$ (1 for E_0, E_1, E_2 , $3/2$ for E_3 and 3 for E_4).

The dual graph $G(X')$ of the Hirzebruch-Jung good resolution $\rho' : (X', E_{X'}) \rightarrow (X, 0)$ of ϕ is in Figure 9. We represent the subgraph $A(X')$ by a double-line joining the arrows associated to the strict transforms of $\{f = 0\}$ and $\{g = 0\}$.

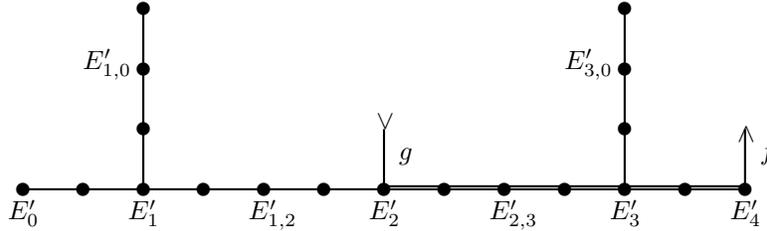


FIGURE 9. Graph of the Hirzebruch-Jung resolution of ϕ

The dual graph $G(\tilde{X})$ of the minimal good resolution $\rho : (\tilde{X}, E_{\tilde{X}}) \rightarrow (X, 0)$ of ϕ , is obtained from the one in Figure 9 by blowing-down E'_0, E'_4 and 4 other vertices of self-intersection -1: $E'_{1,2}, E'_{2,3}, E'_{1,0}, E'_{3,0}$.

In $G(\tilde{X})$ each irreducible component of the exceptional divisor associated to the vertices of $G(\tilde{X})$ is of genus zero and of self-intersection equal to -2, except the one intersected by the strict transform of $\{g = 0\}$ which has self-intersection -3. We weight $G(\tilde{X})$ with the Hironaka quotients of $(f = u \circ \phi, g = v \circ \phi)$. It is represented in Figure 10.

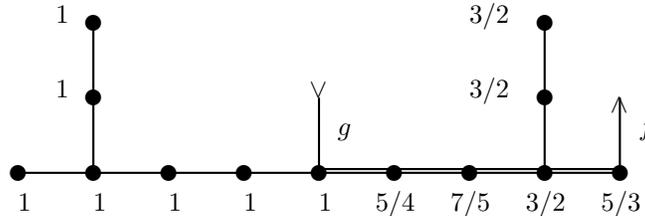


FIGURE 10. Graph of the minimal resolution of ϕ

In this example $A(\tilde{X})$ (respectively $A(X')$) is strictly included in $G(\tilde{X})$ (respectively $G(X')$). Moreover $\overline{G(\tilde{X}) \setminus A(\tilde{X})}$ (respectively $\overline{G(X') \setminus A(X')}$) admits two connected components on which the Hironaka quotients are respectively equal to 1 and $3/2$. These two connected components represent the two rupture zones. So the strict transform of the polar curve meets the exceptional divisor $E_{\tilde{X}}$ in each zone represented by the connected components of $\overline{G(\tilde{X}) \setminus A(\tilde{X})}$.

8.3. **Example 3.** Let us consider the surface $(X, 0)$ of the following equation:

$$z^2 = (x + y^3)(x + y^2)(y^{34} - x^{13}).$$

- (1) Let us first consider the case where $\phi_1 : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is the projection on the $(x + y, y)$ -plane. It is a generic projection.

The discriminant locus of $\phi_1 = (f_1, g_1)$ is the curve Δ_1 which admits three components with Puiseux expansions given by :

$$u = v - v^2$$

$$u = v - v^3$$

$$u = v + v^{34/13}$$

Notice that the three components of Δ_1 admit 1 as first Puiseux exponent and respectively 2, 3, $34/13$ as second Puiseux exponent.

The coordinate axes are transverse to the discriminant locus of ϕ_1 . Hence the maximal arc of the tree of the minimal embedded resolution of Δ_1^+ has a unique vertex of Hironaka quotient equal to one. Moreover the Hironaka quotients are constant in the tree $G(Z)$ of the minimal embedded resolution of Δ_1^+ . The dual graph $G(Z)$ is in Figure 11.

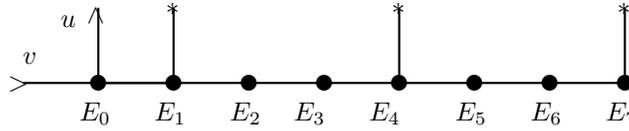


FIGURE 11. Graph of the minimal resolution of Δ_1^+

The Hironaka quotient associated to each irreducible component of E_Z is equal to one. Only the vertex E_0 of $G(Z)$ belongs to $S(Z)$.

The dual graph $G(X')$ of R admits a cycle created by the normalization. The irreducible component E'_9 is obtained by the resolution $\bar{\rho}$. The irreducible components of the exceptional divisor associated to the vertices of $G(X')$ have a genus equal to zero. The subgraph $A(X')$ (resp. $A(\tilde{X})$) only consists of the union of the vertex E'_1 with the two arrows. The subgraph $\overline{G(X') \setminus A(X')}$ (which is $G(X')$ minus the two arrows) is connected and the associated Hironaka quotient is equal to one.

The minimal good resolution ρ is obtained by blowing down E'_6 . Its dual graph is in Figure 13.

In this case there exists a unique rupture zone. So we do not obtain informations on the position of Γ_ϕ in (X, p) . This is due to the fact that the projection ϕ_1 is generic. As explained in [3] we can twist the morphism ϕ_1 in a morphism ϕ_2 such that $\Gamma_{\phi_1} = \Gamma_{\phi_2}$ and the maximal arcs of ϕ_2 are non trivial. In what follows we give such a construction.

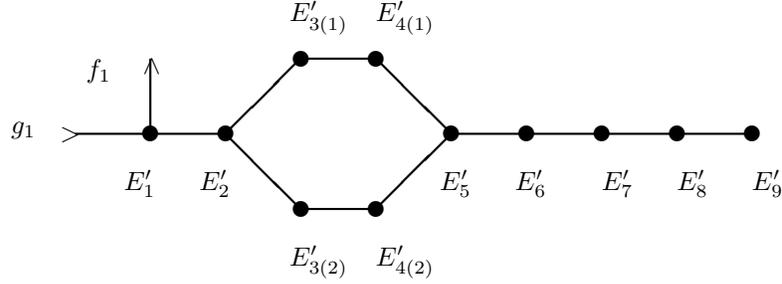


FIGURE 12. The graph of the Hirzebruch-Jung resolution of ϕ_1

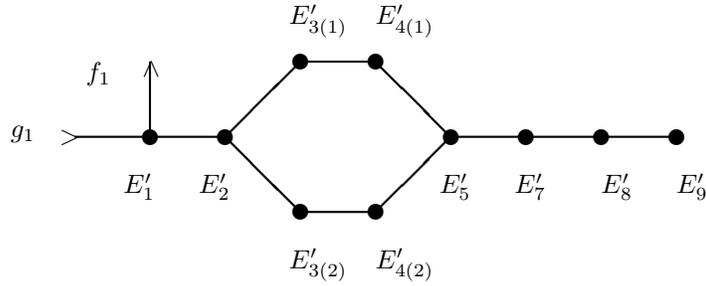


FIGURE 13. The graph of the minimal resolution of ϕ_1

- (2) We take $\phi_2 = (f_2, g_2)$ on the (x, y) -plane. Then ϕ_1 and ϕ_2 have the same critical locus. The discriminant locus of ϕ_2 is the curve $\Delta_2 : (u + v^3)(u + v^2)(v^{34} - u^{13}) = 0$. The first Puiseux exponents of the components of Δ_2 are 2, 3, 34/13. The tree $G(Z)$ of the minimal embedded resolution of Δ_2^+ is in Figure 14.

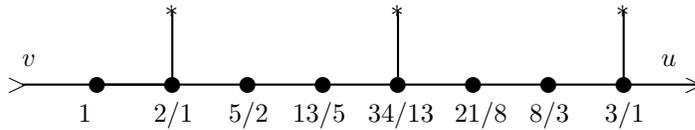


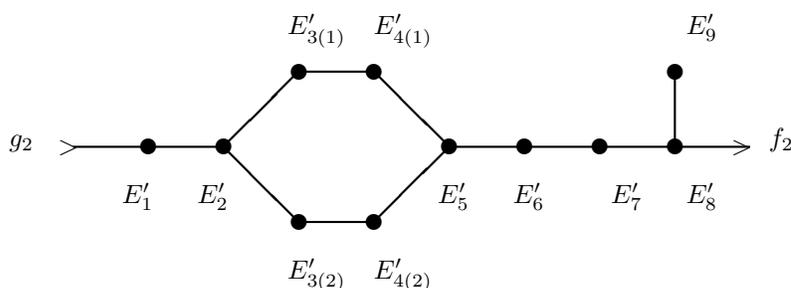
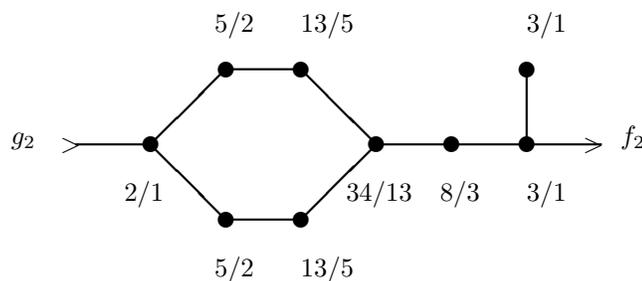
FIGURE 14. Graph of the minimal resolution of Δ_2^+

Each vertex of $G(Z)$ belongs to $S(Z)$.

The subgraph $\overline{G(X') \setminus A(X')}$ admits a unique connected component which only contains the two vertices E'_9 and E'_8 having Hironaka quotient equal to 3.

The graph of the minimal resolution of ϕ_2 is obtained by blowing-down E'_1 and E'_6 .

Here $G(\tilde{X}) \setminus A(\tilde{X})$ has only one connected component. It corresponds to a rupture zone with one rupture component. There are two other rupture zones which are reduced to the other rupture components of $G(\tilde{X})$ (which are contained in $A(\tilde{X})$). Only looking at Figure 16, we see that $\tilde{\Gamma}_{\phi_2}$ has three bunches associated to the three rupture zones.

FIGURE 15. The graph of the Hirzebruch-Jung resolution of ϕ_2 FIGURE 16. Graph of the minimal resolution of ϕ_2

REFERENCES

- [1] W. Barth, C. Peters, A. Van de Ven, *Compact Complex Surfaces*, Ergebnisse der Mathematik, Springer (1984).
- [2] L. Birbrair, W. Neumann, A. Pichon, *The thick-thin decomposition and the bilipschitz classification of normal surface singularities*, Acta Math. (2014) 212, 199-256. DOI: [10.1007/s11511-014-0111-8](https://doi.org/10.1007/s11511-014-0111-8)
- [3] F. Delgado, H. Maugendre, *On the topology of the image by a morphism of plane curve singularities*, Rev Mat Complut (2014) vol. 27, 369-384. DOI: [10.1007/s13163-013-0141-3](https://doi.org/10.1007/s13163-013-0141-3)
- [4] H. Laufer, *Normal two dimensional singularities*, Ann. of Math. Studies **71**, (1971), Princeton Univ. Press.
- [5] D. T. Lê, *Topological use of Polar Curves*, Proc. Symp. Pure Math., **29**, 1975, 507-512.
- [6] D.T. Lê, H. Maugendre, C. Weber, *Geometry of critical loci*, Journal of the L.M.S. **63** (2001), 533-552.
- [7] D.T. Lê, F. Michel, C. Weber, *Courbes polaires et topologie des courbes planes*, Ann. Scien. Ec. Norm. Sup. **24** (1991), 141-169. DOI: [10.24033/asens.1623](https://doi.org/10.24033/asens.1623)
- [8] D.T. Lê, C. Weber, *Résoudre est un jeu d'enfants*, Sem. Inst. de Estud. con Ibero-america y Portugal, Tordesillas (1998).
- [9] H. Maugendre, *Discriminant of a germ $\Phi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ and Seifert fibered manifolds*, Journal of the L.M.S. **59** (1) (1999), 207-226.
- [10] H. Maugendre, *Discriminant d'un germe $(g, f) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ et quotients de contact dans la résolution minimale de $f \cdot g$* , Annales de la Faculté des Sciences de Toulouse, vol. VII, **3**, 1998, 497-525
- [11] M. Merle, *Invariants polaires des courbes planes*, Inv. Math., **41**, 1977, 103-111. DOI: [10.1007/bf01418370](https://doi.org/10.1007/bf01418370)
- [12] F. Michel, *Jacobian curves for normal complex surfaces*, Brasselet, J-P. (ed.) et al., Singularities II. Geometric and topological aspects. Proceedings of the international conference "School and workshop on the geometry and topology of singularities" in honor of the 60th birthday of Lê Dũng Trùng, Cuernavaca, Mexico, January 8-26, 2007. Providence, RI: American Mathematical Society (AMS). Contemporary Mathematics 475, 135-150 (2008).
- [13] F. Michel, C. Weber, *Topologie des germes de courbes planes*, prépublication de l'université de Genève, (1985).
- [14] W. Neumann, A. Pichon, *Lipschitz geometry of complex surfaces: analytic invariants and equisingularity*, (2012), [arxiv: 1211.4897v1](https://arxiv.org/abs/1211.4897v1)

- [15] P. Popescu-Pampu, *Introduction to Jung's method of resolution of singularities*, in *Topology of Algebraic Varieties and Singularities*. Proceedings of the conference in honor of the 60th birthday of Anatoly Libgober, J. I. Cogolludo-Agustin et E. Hironaka eds. Contemporary Mathematics 538, AMS, 2011, 401-432.
[DOI: 10.1090/conm/538/10616](https://doi.org/10.1090/conm/538/10616)
- [16] B. Teissier, *Introduction to equisingularity problems*, Proc. Symp. Pure Math. **29** (1975), 593-632.

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