
A MCKAY CORRESPONDENCE FOR THE POINCARÉ SERIES OF SOME FINITE SUBGROUPS OF $SL_3(\mathbb{C})$

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Dedicated to the memory of Egbert Brieskorn with great admiration

ABSTRACT. A finite subgroup of $SL_2(\mathbb{C})$ defines a (Kleinian) rational surface singularity. The McKay correspondence yields a relation between the Poincaré series of the algebra of invariants of such a group and the characteristic polynomials of certain Coxeter elements determined by the corresponding singularity. Here we consider some non-abelian finite subgroups G of $SL_3(\mathbb{C})$. They define non-isolated three-dimensional Gorenstein quotient singularities. We consider suitable hyperplane sections of such singularities which are Kleinian or Fuchsian surface singularities. We show that we obtain a similar relation between the group G and the corresponding surface singularity.

INTRODUCTION

In [E4] we showed that the Poincaré series of the coordinate algebra of a two-dimensional quasihomogeneous singularity is the quotient of two polynomials one of which is related to the characteristic polynomial of the monodromy of the singularity. There are two special cases of this result. One is the case of a Kleinian singularity not of type A_{2n} . The Kleinian singularities are defined by quotients of \mathbb{C}^2 by finite subgroups of $SL_2(\mathbb{C})$. In this case, the relation means that the Poincaré series is the quotient of the characteristic polynomials of the Coxeter element and the affine Coxeter element of the corresponding root system of type ADE. We derived this relation from the McKay correspondence. The other case is the case of a Fuchsian singularity. A Fuchsian singularity is defined by the action of a Fuchsian group (of the first kind) $\Gamma \subset PSL(2, \mathbb{R})$ on the tangent bundle $T_{\mathbb{H}}$ of the upper half plane \mathbb{H} . For a Fuchsian hypersurface singularity (or more generally for a Fuchsian singularity of genus 0 [EP]), we showed that the Poincaré series is the quotient of two characteristic polynomials of Coxeter elements [E5].

Here we consider a similar relation for the Poincaré series of some non-abelian finite subgroups of $SL_3(\mathbb{C})$. The non-abelian finite subgroups of $SL_3(\mathbb{C})$ define non-isolated three-dimensional Gorenstein quotient singularities. We consider those groups where the natural three-dimensional representation is irreducible and the corresponding quotient singularity has a certain hyperplane section which is a Kleinian or Fuchsian singularity. We show, that in this way, we again obtain relations between the Poincaré series of the algebra of invariants of the group and the characteristic polynomials of certain Coxeter elements determined by the corresponding Kleinian or Fuchsian singularity.

The famous paper [Br] of E. Brieskorn is fundamental for the study of Kleinian singularities. The Kleinian singularities were a central theme in Brieskorn's research and we owe Brieskorn many beautiful and important results about these singularities. Therefore I would like to express my great admiration for him in dedicating this paper to his memory.

G	$ G $	x, y, z	c_G	$R(x, y, z)$	Sing.	$\alpha_1, \dots, \alpha_m$
\mathcal{C}_{2n+1}	$2n+1$	$2, 2n+1, 2n+1$	1	$x^{2n+1} + y^2 + z^2$	A_{2n}	$2n$
\mathcal{C}_{2n}	$2n$	$2, 2n, 2n$	2	$x^{2n} + y^2 + z^2$	A_{2n-1}	$2n-1$
\mathcal{D}_n	$4n$	$4, 2n, 2n+2$	2	$x^{n+1} + xy^2 + z^2$	D_{n+2}	$2, 2, n$
\mathcal{T}	24	6, 8, 12	2	$x^4 + y^3 + z^2$	E_6	2, 3, 3
\mathcal{O}	48	8, 12, 18	2	$x^3y + y^3 + z^2$	E_7	2, 3, 4
\mathcal{I}	120	12, 20, 30	2	$x^5 + y^3 + z^2$	E_8	2, 3, 5

TABLE 1. Subgroups of $\mathrm{SL}_2(\mathbb{C})$ and surface singularities1. FINITE SUBGROUPS OF $\mathrm{SL}_2(\mathbb{C})$ AND $\mathrm{SL}_3(\mathbb{C})$ AND NORMAL SURFACE SINGULARITIES

Let G be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. The classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ up to linear equivalence is well-known, see e.g. [Kl]. There are up to conjugacy five classes of such groups: the cyclic groups \mathcal{C}_ℓ , the binary dihedral groups \mathcal{D}_n , the binary tetrahedral group \mathcal{T} , the binary octahedral group \mathcal{O} , and the binary icosahedral group \mathcal{I} . The quotients of \mathbb{C}^2 by these groups were studied by E. Brieskorn [Br]. Equations for these singularities can be obtained from generators and relations of the algebra of invariant polynomials with respect to G . This algebra has three generators x, y, z in each case which satisfy an equation $R(x, y, z) = 0$. The degrees of the generators and the equation $R(x, y, z) = 0$ are indicated in Table 1. (They can be found, e.g., in [Sp].) The equations define isolated hypersurface singularities in \mathbb{C}^3 , the so called *Kleinian singularities*.

The finite subgroups of $\mathrm{SL}_3(\mathbb{C})$ were classified up to linear equivalence by H. F. Blichfeldt, G. A. Miller, and L. E. Dickson [Bl, MBD] with two missing cases (see [YY]). There are 12 types of finite subgroups of $\mathrm{SL}_3(\mathbb{C})$: (A)–(L). There are four infinite series (A)–(D). The groups of type (A) are the diagonal abelian groups and the groups of type (B) are isomorphic to transitive finite subgroups of $\mathrm{GL}_2(\mathbb{C})$. Here the natural 3-dimensional representation is not irreducible. Type (C) is the infinite series $\Delta(3n^2)$ of groups and type (D) the series $\Delta(6n^2)$ (for the notation see [HH, LNR, EL]). Moreover, we have 8 exceptional subgroups (E)–(L).

We consider those subgroups of type (C)–(L) which admit a certain hyperplane section which defines a Kleinian or Fuchsian singularity. Generators and relations for the algebra of invariant polynomials with respect to G have been computed in [YY] (see also [We] for some cases). They correspond to non-isolated Gorenstein quotient singularities \mathbb{C}^3/G . These singularities are either hypersurface singularities in \mathbb{C}^4 or complete intersection singularities in \mathbb{C}^5 . We denote the coordinates of these spaces by w, x, y, z and w, x, y, z, u respectively. We consider hyperplane sections of these singularities, namely we consider the restrictions of the equations to the hyperplane $w = 0$. For the series (C) and (D) the hyperplane sections of the corresponding singularities for the first few elements of these series are listed in Table 2. It turns out that the singularities corresponding to the series (C) ($\Delta(3n^2)$) belong to Arnold's E -series whereas those of type (D) ($\Delta(6n^2)$) belong to Arnold's Z -series (n even) or are complete intersection singularities (n odd) (for the definition of these series see [Arn]). The subgroups which correspond to Kleinian singularities are the tetrahedral group $T = \Delta(3 \cdot 2^2)$ and the octahedral group $O = \Delta(6 \cdot 2^2)$ which correspond to the Kleinian singularities E_6 and E_7 respectively. Those which correspond to Fuchsian singularities are $\Delta(3 \cdot 4^2)$ (E_{14}), $\Delta(6 \cdot 4^2)$ (Z_{11}), $\Delta(6 \cdot 6^2)$ ($Z_{1,0}$), and $\Delta(6 \cdot 3^2)$ which corresponds to the elliptic complete intersection singularity $\delta 1$ in C. T. C. Wall's notation [Wa2]. (For a list of Fuchsian hypersurface and complete intersection singularities see [E5].) These are 6 cases. The remaining 8 exceptional subgroups of types (E)–(L) all correspond to Fuchsian singularities except in the case (H) which is the icosahedral group I corresponding

G	$ G $	$w, x, y, z, (u)$	c_G	$R(0, x, y, z, (u))$	Sing.
$\Delta(3 \cdot 2^2)$	12	2, 3, 4, 6	1	$z^2 + 4y^3 + 27x^4$	E_6
$\Delta(3 \cdot 3^2)$	27	3, 3, 6, 9	3	$z^2 + 4y^3 + 27x^6$	\tilde{E}_8
$\Delta(3 \cdot 4^2)$	48	4, 3, 8, 12	1	$z^2 + 4y^3 + 27x^8$	E_{14}
$\Delta(3 \cdot 5^2)$	75	5, 3, 10, 15; 30	1	$z^2 + 4y^3 + 27x^{10}$	E_{18}
$\Delta(6 \cdot 2^2)$	24	2, 4, 6, 9	1	$z^2 + 4xy^3 + 27x^3$	E_7
$\Delta(6 \cdot 3^2)$	54	6, 6, 6, 6, 9	3	$\begin{cases} z^2 - xy \\ u^2 + 4xyz + 27x^3 \end{cases}$	$\delta 1$
$\Delta(6 \cdot 4^2)$	96	4, 6, 8, 15	1	$z^2 + 4xy^3 + 27x^5$	Z_{11}
$\Delta(6 \cdot 5^2)$	150	10, 6, 8, 10, 15	1	$\begin{cases} z^2 - xy \\ u^2 + 4x^2yz + 27x^5 \end{cases}$	no name
$\Delta(6 \cdot 6^2)$	216	6, 6, 12, 21	3	$z^2 + 4xy^3 + 27x^7$	$Z_{1,0}$
$\Delta(6 \cdot 7^2)$	294	14, 6, 10, 14, 21	1	$\begin{cases} z^2 - xy \\ u^2 + 4x^3yz + 27x^7 \end{cases}$	no name
$\Delta(6 \cdot 8^2)$	384	8, 6, 16, 27; 54	1	$z^2 + 4xy^3 + 27x^9$	Z_{19}

TABLE 2. The first subgroups of types (C) and (D) and surface singularities

G	$ G $	$w, x, y, z, (u)$	c_G	$R(0, x, y, z, (u))$	Sing.
(C): T	12	2, 3, 4, 6	1	$z^2 + 4y^3 + 27x^4$	E_6
$\Delta(3 \cdot 4^2)$	48	4, 3, 8, 12	1	$z^2 + 4y^3 + 27x^8$	E_{14}
(D): O	24	2, 4, 6, 9	1	$z^2 + 4xy^3 + 27x^3$	E_7
$\Delta(6 \cdot 3^2)$	54	6, 6, 6, 6, 9	3	$\begin{cases} z^2 - xy \\ u^2 + 4xyz + 27x^3 \end{cases}$	$\delta 1$
$\Delta(6 \cdot 4^2)$	96	4, 6, 8, 15	1	$z^2 + 4xy^3 + 27x^5$	Z_{11}
$\Delta(6 \cdot 6^2)$	216	6, 6, 12, 21	3	$z^2 + 4xy^3 + 27x^7$	$Z_{1,0}$
(E)	108	6, 6, 9, 12, 12	3	$\begin{cases} 9u^2 - 12z^2 \\ 432y^2 - x^3 - 36xz \end{cases}$	$K'_{1,0}$
(F)	216	6, 9, 12, 12	3	$4z^3 - 144yz^2 + 1728y^2z - 186624x^4$	U_{12}
(G)	648	9, 12, 18, 18	6	$4z^3 - 9yz^2 + 6y^2z - y^3 + 6912x^3y$	$U_{1,0}$
(H)= I	60	2, 6, 10, 15	1	$z^2 - y^3 + 1728x^5$	E_8
(I)	168	4, 6, 14, 21	1	$z^2 - y^3 - 1728x^7$	E_{12}
(J)	180	6, 6, 12, 15	3	$y^3 - xz^2 + 64x^2y^2$	$Q_{2,0}$
(K)	504	6, 12, 18, 21	3	$y^3 - xz^2 - 256x^3y$	Q_{11}
(L)	1080	6, 12, 30, 45	3	$459165024z^2 - 25509168y^3 - (7558272 - 2519424\sqrt{15}i)x^5y$	E_{13}

TABLE 3. Subgroups of $SL_3(\mathbb{C})$ and surface singularities

to the Kleinian singularity E_8 . Altogether we have 14 cases which we will consider in this paper. They are listed in Table 3. These singularities are surface singularities and they are isolated except in the three cases $\Delta(6 \cdot 3^2)$, (E) and (J). They correspond to Kleinian singularities in the cases T , O and (H) (the icosahedral group I) and to Fuchsian singularities in the other cases. They correspond to simple (T , O , I), unimodal ($\Delta(3 \cdot 4^2)$, $\Delta(6 \cdot 4^2)$, (F), (I), (K), (L)) and bimodal ($\Delta(6 \cdot 6^2)$, (G), (J)) hypersurface singularities, to the unimodal complete intersection singularity of type $K'_{1,0}$ (type (E)) in Wall's notation [Wal], and to the elliptic

complete intersection singularity $\delta 1$. The names of the hypersurface singularities according to V. I. Arnold’s classification [Arn] are indicated in the last column of Table 3.

2. POINCARÉ SERIES AND COXETER ELEMENTS

We now consider the isolated singularities corresponding to these singularities. They are quasihomogeneous. This means the following. A complex polynomial $f(x_1, \dots, x_n)$ is called quasihomogeneous, if there are positive integers w_1, \dots, w_n (called *weights*) and d (called *degree*) such that $f(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda^d f(x_1, \dots, x_n)$ for $\lambda \in \mathbb{C}^*$. A complete intersection singularity given as the zero set of polynomials $f_1(x_1, \dots, x_n), \dots, f_k(x_1, \dots, x_n)$ is called quasihomogeneous, if f_1, \dots, f_k are quasihomogeneous with respect to the same weights w_1, \dots, w_n but degrees d_1, \dots, d_k respectively. We call the system $W := (w_1, \dots, w_n; d_1, \dots, d_k)$ the *weight system* corresponding to the set of polynomials. Let c_W be the greatest common divisor of $w_1, \dots, w_n, d_1, \dots, d_k$. The weight system is called *reduced* if $c_W = 1$.

We assume that $f_1(0) = \dots = f_k(0) = 0$ and the system of equations $f_1 = \dots = f_k = 0$ has an isolated singularity at the origin. The coordinate algebra $A_f := \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_k)$ is a \mathbb{Z} -graded algebra with respect to the system of weights $(w_1, \dots, w_n; d_1, \dots, d_k)$. Therefore we can consider the decomposition of A_f as a \mathbb{Z} -graded \mathbb{C} -vector space:

$$A_f := \bigoplus_{k=0}^{\infty} A_{f,k}, \quad A_{f,k} := \{g \in A_f \mid g(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda^k g(x_1, \dots, x_n)\}.$$

The formal power series $p_f(t) := \sum_{k=0}^{\infty} (\dim_{\mathbb{C}} A_{f,k}) t^k$ is called the *Poincaré series* of A_f . It is given by

$$p_f(t) = \frac{\prod_{i=1}^k (1 - t^{d_i})}{\prod_{j=1}^n (1 - t^{w_j})}.$$

Let (X, x) be a Kleinian singularity. Then the minimal resolution of the singularity x has an exceptional divisor with the dual graph depicted in Fig. 1 with $m = 1$ in the case of the A_n -singularities and $m = 3$ in the other cases (see, e.g., [Br]). Here all vertices correspond to rational curves of self-intersection number -2 , the mutual intersection numbers are either 0 or 1, and two vertices are joined by an edge if and only if the intersection number of the corresponding rational curves is equal to 1. The values of the numbers $\alpha_1, \dots, \alpha_m$ are indicated in Table 1. They are the *Dolgachev numbers* of the singularity, see [ET]. It turns out that these graphs are precisely the ordinary Coxeter-Dynkin diagrams of type ADE. (Note that the corresponding intersection matrix is the Cartan matrix multiplied by -1 .)

Now let (X, x) be a Fuchsian hypersurface or complete intersection singularity. A natural compactification of X is given by $\bar{X} := \text{Proj}(A_f[t])$, where t has degree 1 for the grading of $A_f[t]$ (see [P]). This is a normal projective surface with a \mathbb{C}^* -action. The surface \bar{X} may acquire additional singularities on the boundary $\bar{X}_{\infty} := \bar{X} \setminus X = \text{Proj}(A_f)$. Let $g = g(\bar{X}_{\infty})$ be the genus of the boundary. We assume $g = 0$. Let $\pi : S \rightarrow \bar{X}$ be the minimal normal crossing resolution of all singularities of \bar{X} . The preimage \tilde{X}_{∞} of \bar{X}_{∞} under $\pi : S \rightarrow \bar{X}$ consists of the strict transform δ_0 of \bar{X}_{∞} and m chains $\delta_1^i, \dots, \delta_{\alpha_i-1}^i$, $i = 1, \dots, m$, of rational curves of self-intersection -2 which intersect again according to the dual graph shown in Figure 1 (see, e.g., [D, E5]). By the adjunction formula and $g = 0$, the self-intersection number of the rational curve δ_0 is also -2 . The numbers $\alpha_1, \dots, \alpha_m$ of the Fuchsian singularities corresponding to finite subgroups of $\text{SL}_3(\mathbb{C})$ are indicated in Table 4. They are again the *Dolgachev numbers* of the singularity, see [ET, E3].

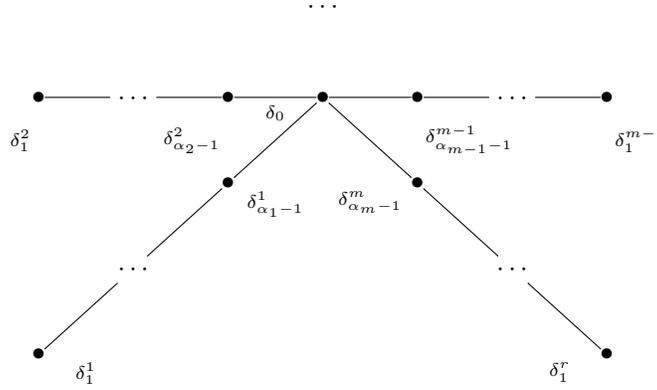


FIGURE 1. The graph $T_{\alpha_1, \dots, \alpha_m}^-$

G	Name	Normal form	Weights	$\alpha_1, \dots, \alpha_m$
(C): T	E_6	$z^2 + y^3 + x^4$	3,4,6;12	2, 3, 3
$\Delta(3 \cdot 4^2)$	E_{14}	$z^2 + y^3 + x^8$	3,8,12;24	3, 3, 4
(D): O	E_7	$z^2 + y^3 + yx^3$	4,6,9;18	2, 3, 4
$\Delta(6 \cdot 3^2)$	δ_1	$\left\{ \begin{array}{l} xy + z^2 \\ x^3 + y^3 + z^3 + w^2 \end{array} \right\}$	2,2,2,3;4,6	2, 2, 2, 2, 2
$\Delta(6 \cdot 4^2)$	Z_{11}	$z^2 + xy^3 + x^5$	6,8,15;30	2, 3, 8
$\Delta(6 \cdot 6^2)$	$Z_{1,0}$	$z^2 + xy^3 + x^7$	2,4,7;14	2, 2, 2, 4
(E)	$K'_{1,0}$	$\left\{ \begin{array}{l} xu + y^2 \\ ax^4 + xy^2 + z^2 + u^2, \\ a \neq 0, \frac{1}{4} \end{array} \right\}$	2,3,4,4;6,8	2, 2, 4, 4
(F)	U_{12}	$z^3 + y^3 + x^4$	3,4,4;12	4, 4, 4
(G)	$U_{1,0}$	$z^3 + yz^2 + x^3y$	2,3,3;9	2, 3, 3, 3
(H)=I	E_8	$z^2 + y^3 + x^5$	6,10,15;30	2, 3, 5
(I)	E_{12}	$z^2 + y^3 + x^7$	6,14,21;42	2, 3, 7
(J)	$Q_{2,0}$	$xz^2 + y^3 + x^4y$	2,4,5;12	2, 2, 2, 5
(K)	Q_{11}	$xz^2 + y^3 + yx^3$	4,6,7;18	2, 4, 7
(L)	E_{13}	$z^2 + y^3 + x^5y$	4,10,15;30	2, 4, 5

TABLE 4. Normal forms, reduced weight systems, and Dolgachev numbers

We call the graph $T_{\alpha_1, \dots, \alpha_m}^-$ a *Coxeter-Dynkin diagram*. Let V_- be the free \mathbb{Z} -module with the basis

$$\delta_1^1, \dots, \delta_{\alpha_1-1}^1, \delta_1^2, \dots, \delta_{\alpha_2-1}^2, \dots, \delta_1^m, \dots, \delta_{\alpha_m-1}^m, \delta_0$$

equipped with the symmetric bilinear form $\langle -, - \rangle$ given by the intersection matrix corresponding to Fig. 1. This defines a lattice $(V_-, \langle -, - \rangle)$.

We consider two extensions of this lattice. Let $V_0 = V_- \oplus \mathbb{Z}\delta_1$ with the symmetric bilinear form defined by Fig. 2. Here the double dashed line between δ_0 and δ_1 means $\langle \delta_0, \delta_1 \rangle = -2$. Let $V_+ = V_0 \oplus \mathbb{Z}\delta_2$ with the symmetric bilinear form defined by Fig. 3.

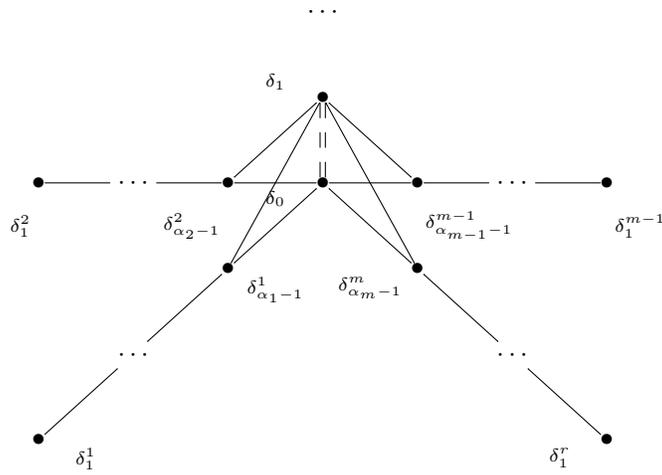


FIGURE 2. The graph $T_{\alpha_1, \dots, \alpha_m}$

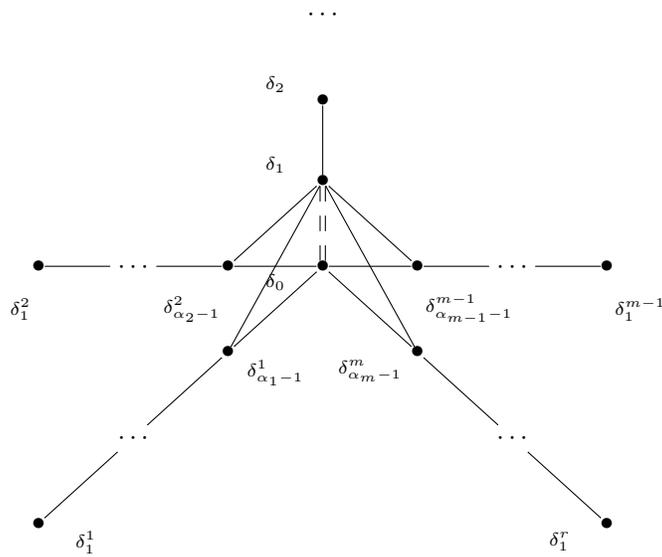


FIGURE 3. The graph $T_{\alpha_1, \dots, \alpha_m}^+$

If $(V, \langle -, - \rangle)$ is an arbitrary lattice and $e \in V$ is a *root*, i.e. $\langle e, e \rangle = -2$, then the reflection corresponding to e is defined by

$$s_e(x) = x - \frac{2\langle x, e \rangle}{\langle e, e \rangle} e = x + \langle x, e \rangle e \quad \text{for } x \in V.$$

If $B = (e_1, \dots, e_n)$ is an ordered basis consisting of roots, then the *Coxeter element* τ corresponding to B is defined by

$$\tau = s_{e_1} s_{e_2} \cdots s_{e_n}.$$

For a Coxeter element τ , let $\Delta(t) = \det(1 - \tau^{-1}t)$ be its characteristic polynomial, using a suitable normalization.

If D is a Coxeter-Dynkin diagram, then we denote by $\Delta_D(t)$ the characteristic polynomial of the Coxeter element corresponding to the graph D . These polynomials can be computed as in [E1] and one gets

$$\begin{aligned} \Delta_{T_{\alpha_1, \dots, \alpha_m}^-}(t) &= (1+t) \prod_{i=1}^m \frac{1-t^{\alpha_i}}{1-t} - t \sum_{i=1}^m \frac{1-t^{\alpha_i-1}}{1-t} \prod_{\substack{j=1 \\ j \neq i}}^m \frac{1-t^{\alpha_j}}{1-t}, \\ \Delta_{T_{\alpha_1, \dots, \alpha_m}}(t) &= (1-t)^{2-m} (1-t^{\alpha_1}) \cdots (1-t^{\alpha_m}), \\ \Delta_{T_{\alpha_1, \dots, \alpha_m}^+}(t) &= (1-2t-2t^2+t^3) \prod_{i=1}^m \frac{1-t^{\alpha_i}}{1-t} + t^2 \sum_{i=1}^m \frac{1-t^{\alpha_i-1}}{1-t} \prod_{\substack{j=1 \\ j \neq i}}^m \frac{1-t^{\alpha_j}}{1-t}. \end{aligned}$$

(The last two formulas can also be found in [E2, p. 98], but note that, unfortunately, there is a misprint in [E2, p. 98].)

Now we can state the main result of [EP].

Theorem 1. (i) *For a Kleinian singularity not of type A_{2n} we have*

$$p_f(t) = \frac{\Delta_{T_{\alpha_1, \dots, \alpha_m}^-}(t)}{\Delta_{T_{\alpha_1, \dots, \alpha_m}}(t)}.$$

(ii) *For a Fuchsian singularity with $g = 0$ we have*

$$p_f(t) = \frac{\Delta_{T_{\alpha_1, \dots, \alpha_m}^+}(t)}{\Delta_{T_{\alpha_1, \dots, \alpha_m}}(t)}.$$

Remark 2. Unfortunately, the exclusion of the case A_{2n} is only implicit in [EP] and was forgotten in the statement of [EP, Theorem 1].

Remark 3. Note that we have $T_{2,3,3} \sim T_{3,3,3}^-$, $T_{2,3,4} \sim T_{2,4,4}^-$, $T_{2,3,5} \sim T_{2,3,6}^-$, where \sim means equivalence under the braid group action, see [E6]. Similarly, one can show that the graphs T_{2n-1} , $n \geq 1$, and $T_{2,2,n}$, $n \geq 2$, are equivalent under the braid group action to the extended Coxeter-Dynkin diagrams of type A_{2n-1} and D_{n+2} respectively.

3. POINCARÉ SERIES OF SUBGROUPS OF $SL_2(\mathbb{C})$ AND $SL_3(\mathbb{C})$

Let G be a finite subgroup of $SL_n(\mathbb{C})$ for $n = 2, 3$. Consider the algebra of complex polynomials $\mathbb{C}[x_1, \dots, x_n]$ graded by the degree for homogeneous ones. It is isomorphic to the symmetric algebra

$$S := S(\mathbb{C}^n) = \bigoplus_{k=0}^{\infty} S^k(\mathbb{C}^n),$$

where $S^k(\mathbb{C}^n)$ denotes the k -th symmetric power of \mathbb{C}^n . Let S^G be the algebra of invariant polynomials with respect to G .

For $n = 2$, it is generated by 3 elements x, y, z which satisfy a relation $R(x, y, z) = 0$. The elements x, y, z correspond to invariant polynomials and their degrees correspond to the weights of these variables. Let c_G denote the greatest common divisor of these weights. The weights of the variables x, y, z , the number c_G , and the polynomial $R(x, y, z)$ are indicated in Table 1.

Now let G be one of the finite subgroups of $SL_3(\mathbb{C})$ of Table 3. Except in the cases (E) and $\Delta(6 \cdot 3^2)$, the algebra S^G is generated by 4 elements w, x, y, z which satisfy a relation $R(w, x, y, z)$. In the cases (E) and $\Delta(6 \cdot 3^2)$, S^G is generated by 5 elements w, x, y, z, u which

satisfy two relations $R_1(w, x, y, z, u) = 0$ and $R_2(w, x, y, z, u) = 0$. The degrees of the invariants and the polynomials $R(w, x, y, z)$ and $R_1(w, x, y, z, u), R_2(w, x, y, z, u)$ respectively can be found in [YY]. The degrees of the invariant polynomials and the restriction to the hyperplane $w = 0$ of the polynomials $R(w, x, y, z)$ and $R_1(w, x, y, z, u), R_2(w, x, y, z, u)$ respectively are indicated in Table 3. Let c_G be the greatest common divisor of the weights of the remaining variables $x, y, z, (u)$ (with the weight of w excluded). The number c_G is also indicated in Table 3. Note that, except in the case (G), c_G also divides the weight of w .

For $n = 2$, the algebra $A_G := S^G = \mathbb{C}[x, y, z]/R(x, y, z)$ coincides with the coordinate algebra A_f of the corresponding singularity indicated in the last column of Table 1 up to the grading. The grading is shifted by c_G . For $n = 3$ and G one of the cases of Table 3 except the cases (E) and $\Delta(6 \cdot 3^2)$, the algebra $A_G := \mathbb{C}[x, y, z]/R(0, x, y, z)$ coincides with the coordinate algebra A_f of the corresponding singularity indicated in the last column of Table 3 with the grading shifted by c_G . In the cases (E) and $\Delta(6 \cdot 3^2)$, the algebra $A_G := \mathbb{C}[x, y, z, u]/(R_1(0, x, y, z, u), R_2(0, x, y, z, u))$ coincides with the coordinate algebra A_f of the complete intersection singularity $K'_{1,0}$ and $\delta 1$ respectively, again with the grading shifted by c_G . Let $p_G(t)$ be the Poincaré series of the algebra of A_G . Then we have

$$p_G(t) = p_f(t^{c_G}) \text{ for } G \subset \text{SL}_2(\mathbb{C}), \quad p_G(t) = \frac{p_f(t^{c_G})}{(1 - t^{\deg w})} \text{ for } G \subset \text{SL}_3(\mathbb{C}).$$

The finite subgroups $G \subset \text{SL}_n(\mathbb{C})$ for $n = 2, 3$ under consideration have a natural n -dimensional representation γ which is irreducible (except in the cases $G = \mathcal{C}_l$). Let γ^* be its contragredient representation. Let $\gamma_0, \dots, \gamma_l$ be the irreducible representations of G , where γ_0 is the trivial representation. Let $B = (b_{ij})$ and $B^* = (b_{ij}^*)$ be the $(l + 1) \times (l + 1)$ -matrices defined by decomposing the tensor products

$$\gamma_j \otimes \gamma = \bigoplus_i b_{ij} \gamma_i \quad \text{and} \quad \gamma_j \otimes \gamma^* = \bigoplus_i b_{ij}^* \gamma_i$$

respectively into irreducible components.

For each integer $k \geq 0$, let ρ_k be the representation of G on $S^k(\mathbb{C}^n)$ induced by its natural action on \mathbb{C}^n . We have a decomposition $\rho_k = \sum_{i=0}^l v_{ki} \gamma_i$ with $v_{ki} \in \mathbb{Z}$. We associate to ρ_k the vector $v_m = (v_{m0}, \dots, v_{ml})^t \in \mathbb{Z}^{l+1}$. As in [K, p. 211] we define

$$P_G(t) := \sum_{m=0}^{\infty} v_m t^m.$$

This is a formal power series with coefficients in \mathbb{Z}^{l+1} . We also put $P_G(t)_i := \sum_{m=0}^{\infty} v_{mi} t^m$. Note that $P_G(t)_0$ is the usual Poincaré series $p_G(t)$ of the group G . Let V denote the set of all formal power series $x = \sum_{m=0}^{\infty} x_m t^m$ with $x_m \in \mathbb{Z}^{l+1}$. This is a free module of rank $l + 1$ over the ring R of formal power series with integer coefficients.

Now let $n = 2$ and $G \subset \text{SL}_2(\mathbb{C})$ be a finite subgroup not of type \mathcal{C}_{2n+1} . Then $c_G = 2$ and we have

$$p_f(t^2) = P_G(t)_0.$$

Moreover, we have $\gamma^* = \gamma$ and therefore $B^* = B$. The irreducible representations of $\text{SL}_2(\mathbb{C})$ are of the form ρ_m , m a non-negative integer. The Clebsch-Gordon formula reads in this case

$$\rho_m \otimes \gamma = \rho_{m+1} \oplus \rho_{m-1}$$

setting $\rho_{-1} = 0$ (cf., e.g., [FH, Exercise 11.11]). This yields the equation

$$Bv_m = v_{m+1} + v_{m-1}.$$

Following [K, p. 222], one can easily derive from this equation that $x = P_G(t)$ is a solution of the following linear equation in V :

$$((1 + t^2)I - tB)x = v_0.$$

Let $M(t)$ be the matrix $(1 + t^2)I - tB$ and $M_0(t)$ be the matrix obtained by replacing the first column of $M(t)$ by $v_0 = (1, 0, \dots, 0)^t$. By Cramer's rule we can derive the following theorem [E4, Sect. 3] (see also [St]).

Theorem 4. *For a finite subgroup $G \subset SL_2(\mathbb{C})$ not of type C_{2n+1} we have*

$$p_f(t^2) = P_G(t)_0 = \frac{\det M_0(t)}{\det M(t)} = \frac{\det(t^2I - \tau)}{\det(t^2I - \tau_a)},$$

where τ is the Coxeter element and τ_a the affine Coxeter element of the corresponding root system of type ADE associated to the singularity defined by the equation $f = 0$ with the same name.

Now let $n = 3$ and $G \subset SL_3(\mathbb{C})$ be a finite subgroup. For a pair a, b of non-negative integers, let $\Gamma_{a,b}$ be the unique irreducible, finite-dimensional representation of $SL_3(\mathbb{C})$ of [FH, Theorem 13.1]. By [FH, Proposition 15.25] and [FH, (13.5)], we have for a non-negative integer m (setting $\Gamma_{-1,b} = 0$) the following Clebsch-Gordon formulas:

$$\begin{aligned} \Gamma_{m,0} \otimes \gamma &= \Gamma_{m+1,0} \oplus \Gamma_{m-1,1}, \\ \Gamma_{m,0} \otimes \gamma^* &= \Gamma_{m-1,0} \oplus \Gamma_{m,1}. \end{aligned}$$

Since $\Gamma_{m,0} = \rho_m$, we can derive from these formulas

$$v_{m+2} = Bv_{m+1} - B^*v_m + v_{m-1}.$$

Therefore $x = P_G(t)$ is a solution of the following linear equation in V (see also [BI, BP]):

$$((1 - t^3)I - tB + t^2B^*)x = v_0.$$

Let $M(t)$ be the matrix $(1 - t^3)I - tB + t^2B^*$ and $M_0(t)$ be the matrix obtained by replacing the first column of $M(t)$ by $v_0 = (1, 0, \dots, 0)^t$. Again Cramer's rule yields

$$P_G(t)_0 = \frac{\det M_0(t)}{\det M(t)}.$$

We have the following theorem:

Theorem 5. *Let $G \subset SL_3(\mathbb{C})$ be one of the groups $T, \Delta(3 \cdot 4^2), O, \Delta(6 \cdot 3^2), \Delta(6 \cdot 4^2), \Delta(6 \cdot 6^2), (E), (F), (G), (H)=I, (I), (J), (K),$ or (L) , let c_G be the greatest common divisor of the weights of the variables x, y, z, u , and let $\alpha_1, \dots, \alpha_m$ be the Dolgachev numbers of the singularity corresponding to G . Moreover, let $q_{a,b}^{(e)}(t) = (1 - t)^a(1 - t^e)^b$ for $a, b, e \in \mathbb{Z}$.*

(i) *For $G = T, O, I (E_6, E_7, E_8)$ we have $c_G = 1$ and*

$$\det M_0(t) = q_{a,b}^{(2)}(t)\Delta_{T_{\alpha_1, \alpha_2, \alpha_3}^-}(t), \quad \det M(t) = (1 - t)q_{a,b}^{(2)}(t)\Delta_{T_{2, \alpha_1, \alpha_2, \alpha_3}}(t),$$

where $(a, b) = (3, 0), (3, 1), (4, 0)$ respectively.

(ii) *For $G=(I), \Delta(3 \cdot 4^2), \Delta(6 \cdot 4^2) (E_{12}, E_{14}, Z_{11})$ we have $c_G = 1$ and*

$$\det M_0(t) = q_{a,b}^{(4)}(t)\Delta_{T_{\alpha_1, \alpha_2, \alpha_3}^+}(t), \quad \det M(t) = (1 - t)q_{a,b}^{(4)}(t)\Delta_{T_{4, \alpha_1, \alpha_2, \alpha_3}}(t).$$

where $(a, b) = (3, 0), (3, 2), (2, 3)$ respectively.

G	$\det M_0(t)$	$\det M(t)$
T	$(1-t)^3 \Delta_{T_{2,3,3}^-}(t)$	$(1-t)^4 \Delta_{T_{2,2,3,3}}(t)$
$\Delta(3 \cdot 4^2)$	$(1-t)^3(1-t^4)^2 \Delta_{T_{3,3,4}^+}(t)$	$(1-t)^4(1-t^4)^2 \Delta_{T_{3,3,4,4}}(t)$
O	$(1-t)^3(1-t^2) \Delta_{T_{2,3,4}^-}(t)$	$(1-t)^4(1-t^2) \Delta_{T_{2,2,3,4}}(t)$
$\Delta(6 \cdot 3^2)$	$\frac{(1-t^3)^{10}}{(1-t^6)^3} \Delta_{T_{2,2,2,2,2,2}^+}(t^3)$	$\frac{(1-t^3)^9}{(1-t^6)^3} \Delta_{T_{2,2,2,2,2}}(t^3)$
$\Delta(6 \cdot 4^2)$	$(1-t)^2(1-t^4)^3 \Delta_{T_{2,3,8}^+}(t)$	$(1-t)^3(1-t^4)^3 \Delta_{T_{2,3,4,8}}(t)$
$\Delta(6 \cdot 6^2)$	$(1-t^3)^7(1-t^6) \Delta_{T_{2,2,2,4}^+}(t^3)$	$(1-t^3)^8(1-t^6) \Delta_{T_{2,2,2,2,4}}(t^3)$
(E)	$\frac{(1-t^3)^8}{(1-t^6)^3} \Delta_{T_{2,2,4,4}^+}(t^3)$	$\frac{(1-t^3)^9}{(1-t^6)^3} \Delta_{T_{2,2,2,4,4}}(t^3)$
(F)	$\frac{(1-t^3)^7}{(1-t^6)^2} \Delta_{T_{4,4,4}^+}(t^3)$	$\frac{(1-t^3)^8}{(1-t^6)^2} \Delta_{T_{2,4,4,4}}(t^3)$
(G)	$\frac{(1-t^3)^4(1-t^6)(1-t^9)}{(1-t^{18})} \Delta_{T_{2,3,3,3}^+}(t^6)$	$\frac{(1-t^3)^4(1-t^6)(1-t^9)^2}{(1-t^{18})} \Delta_{T_{2,3,3,3}}(t^6)$
(H)=I	$(1-t)^4 \Delta_{T_{2,3,5}^-}(t)$	$(1-t)^5 \Delta_{T_{2,2,3,5}}(t)$
(I)	$(1-t)^3 \Delta_{T_{2,3,7}^+}(t)$	$(1-t)^4 \Delta_{T_{2,3,4,7}}(t)$
(J)	$\frac{(1-t^3)^8}{(1-t^6)^2} \Delta_{T_{2,2,2,5}^+}(t^3)$	$\frac{(1-t^3)^9}{(1-t^6)^2} \Delta_{T_{2,2,2,2,5}}(t^3)$
(K)	$\frac{(1-t^3)^6}{(1-t^6)} \Delta_{T_{2,4,7}^+}(t^3)$	$\frac{(1-t^3)^7}{(1-t^6)} \Delta_{T_{2,2,4,7}}(t^3)$
(L)	$\frac{(1-t^3)^7}{(1-t^6)} \Delta_{T_{2,4,5}^+}(t^3)$	$\frac{(1-t^3)^8}{(1-t^6)} \Delta_{T_{2,2,4,5}}(t^3)$

TABLE 5. Determinants of the matrices $M_0(t)$ and $M(t)$

(iii) For $G=(K), (L), (F), \Delta(6 \cdot 6^2), (J), (E)$ ($Q_{11}, E_{13}, U_{12}, Z_{1,0}, Q_{2,0}, K'_{1,0}$) we have $c_G = 3$ and

$$\det M_0(t) = q_{a,b}^{(2)}(t^3) \Delta_{T_{\alpha_1, \dots, \alpha_m}^+}(t^3), \quad \det M(t) = (1-t^3) q_{a,b}^{(2)}(t^3) \Delta_{T_{2, \alpha_1, \dots, \alpha_m}}(t^3),$$

where $(a, b) = (6, -1), (7, -1), (7, -2), (7, 1), (8, -2), (8, -3)$ respectively.

(iv) For $G = \Delta(6 \cdot 3^2)$ ($\delta 1$) we have $c_G = 3, m = 6, \alpha_i = 2$ for $i = 1, \dots, m$, and

$$\det M_0(t) = (1-t^3) q_{9,-3}^{(2)}(t^3) \Delta_{T_{2, \alpha_2, \dots, \alpha_m}^+}(t^3), \quad \det M(t) = q_{9,-3}^{(2)}(t^3) \Delta_{T_{\alpha_2, \dots, \alpha_m}}(t^3).$$

(v) For $G=(G)$ ($U_{1,0}$) we have $c_G = 6$ and

$$\det M_0(t) = q(t^3) \Delta_{T_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}^+}(t^6), \quad \det M(t) = (1-t^9) q(t^3) \Delta_{T_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}}(t^6),$$

$$\text{where } q(t) = \frac{(1-t)^4(1-t^2)(1-t^3)}{(1-t^6)}.$$

Proof. The character tables of the tetrahedral and icosahedral group are given in [Art]. The character table of the octahedral group can be found, e.g., in [HH]. From these tables, one can calculate the matrices B . The matrices B for the remaining cases are given in [BP]. For the case (D), only one example is treated. More complete results for the cases $\Delta(3n^2)$ and $\Delta(6n^2)$ can be found in [LNR] and [EL] respectively. From these results, one can derive the corresponding matrices B . The proof of the theorem is then obtained by a direct calculation from these matrices using the computer algebra system SINGULAR [DGPS]. \square

The results are summarized in Table 5.

Remark 6. Let G be one of the groups T, O, I . In this case, the matrix B is symmetric and we have $B^* = B$. Therefore

$$M(t) = (1-t^3)I - tB + t^2B^* = (1-t)((1+t+t^2)I - tB).$$

ACKNOWLEDGEMENTS

This paper grew out of a common research project with A. Takahashi. The author is grateful to him for useful discussions. He also would like to thank the anonymous referee for useful comments which helped to improve the paper.

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