Journal *of* **Singularities**
Volume 12 (2015), 164-179

Proc. of Singularities in Geometry and Appl. III, Edinburgh, Scotland, 2013
DOI: 10.5427/jsing.2015.12l

ON THE SMOOTHINGS OF NON-NORMAL ISOLATED SURFACE SINGULARITIES

PATRICK POPESCU-PAMPU

Abstract. We show that isolated surface singularities which are non-normal may have Milnor fibers which are non-diffeomorphic to those of their normalizations. Therefore, non-normal isolated singularities enrich the collection of Stein fillings of links of normal isolated singularities. We conclude with a list of open questions related to this theme.

1. Introduction

Let $(S, 0)$ be a germ of irreducible complex analytic space with isolated singularity. Varchenko [50] proved that there is a well-defined isomorphism class of contact structures on its link (or *boundary*, as we prefer to call it in this paper). Following the terminology introduced in [6], we say that a contact manifold which appears in this way is *Milnor fillable*. We use the same name if we forget the contact structure: namely, an oriented odd-dimensional manifold is Milnor-fillable if and only if it is orientation-preserving diffeomorphic to the boundary of an isolated singularity.

If $(S, 0)$ is *smoothable*, that is, if there exist deformations of it with smooth generic fibers, then there exist representatives of such fibers – the so-called *Milnor fibers* of the deformation – which are Stein fillings of the contact boundary of the singularity. Milnor fibers associated to arbitrary smoothings were mainly studied till now for *normal* surface singularities. When they are rational homology balls, they are used for the operation of *rational blow-down* introduced by Fintushel and Stern [7] and generalized by Stipsicz, Szabó, Wahl [49]. Due to the efforts of several researchers, the *normal* surface singularities which have smoothings whose Milnor fibers are rational homology balls are now completely classified. See [36] and [4] for details on this direction of research.

In another direction, there are results which classify *all* the possible Stein fillings (independently of their homology) up to diffeomorphisms, for special kinds of singularities: Ohta and Ono did this for simple elliptic singularities [32] and simple singularities [33], Lisca for cyclic quotient singularities [23], Bhupal and Ono [3] for the remaining quotient surface singularities.

If $(S, 0)$ is fixed, the existence of a holomorphic versal deformation, proved by Grauert [8], shows that, up to diffeomorphisms, there is only a finite number of Stein fillings of its contact boundary which appear as Milnor fibers of its smoothings. For all the previous classes of singularities, there is also a finite number of Stein fillings and even of strong symplectic fillings. This fact is not general. Ohta and Ono [34] showed that there exist Milnor fillable contact 3-manifolds which admit an infinite number of minimal strong symplectic fillings, pairwise not homotopy equivalent. Later, Akhmedov and Ozbagci [1] proved that there exist Milnor fillable contact 3-manifolds which admit even an infinite number of Stein fillings pairwise non-diffeomorphic, but homeomorphic. Moreover, by varying the contact 3-manifold, the fundamental groups of

2000 *Mathematics Subject Classification.* 14B07 (primary), 32S55.
Key words and phrases. Milnor fibers, Milnor fillable manifolds, non-normal singularities, ruled surfaces, Stein fillings, symplectic fillings, smoothings, simple elliptic singularities.

such fillings exhaust all finitely presented groups. For details on this direction of research, one may consult Ozbagci's survey [35].

For simple singularities (see [33]) and for cyclic quotients (see [31]), all Stein fillings are diffeomorphic to the Milnor fibers of the smoothings of a singularity with the given contact link (in each case there is only one such singularity, up to isomorphisms). By contrast, for *simple elliptic singularities*, there exist Stein fillings of their contact boundary which are not diffeomorphic to a Milnor fiber, but to the total space of their minimal resolution.

For instance, in the case of those simple elliptic singularities which are not smoothable (which means, by a theorem of Pinkham [37], that the exceptional divisor of the minimal resolution is an elliptic curve with self-intersection ≤ -10), there is only one Stein filling, which is diffeomorphic to the total space of the minimal resolution.

We explain here (see Section 5), that *this total space is diffeomorphic to the Milnor fiber of a smoothing of a* non-normal *isolated surface singularity, whose normalization is the given non-smoothable simple elliptic singularity.* We do this by using the simplest technique of construction of smoothings, which was called "*sweeping out the cone by hyperplane sections*" by Pinkham [37]. This has the advantage of showing that those Milnor fibers are in fact diffeomorphic to affine algebraic surfaces.

More generally, the results of Laufer [21] and Bogomolov and de Oliveira [5] show that, for any *normal* surface singularity $(S, 0)$, there is a smoothing of an isolated surface singularity whose Milnor fiber is diffeomorphic to the minimal resolution of $(S, 0)$ (see Proposition 5.8).

We wrote this paper in order to emphasize *the problem of the topological study of the smoothings of non-normal isolated singularities.* Let us mention that Jan Stevens has a manuscript [47] which emphasizes the *algebraic* aspects of the deformation theory of such singularities.

We have in mind as potential readers graduate students specializing either in singularity theory or in contact/symplectic topology, therefore we explain several notions and facts which are well-known to specialists of either field, but maybe not to both.

Let us describe briefly the contents of the various sections. In Section 2 we explain basic facts about normal surface singularities, their resolutions and the classes of rational, minimally elliptic and simple elliptic singularities. In Section 3 we explain the basic notions about deformations needed in the sequel. In Section 4 we explain the technique of sweeping out a cone by hyperplane sections and the reason why one does not necessarily get in this way a normal singularity, even if the starting singularity is normal. In Section 5 we continue with material about very ample curves on ruled surfaces, and we apply it to the construction of the desired smoothings. In the last section, we list a series of open questions which we consider to be basic for the knowledge of the topology of deformations of *isolated non-normal singularities*. By a theorem of Kollár explained in Remark 5.10, the case of *rational* surface singularities is special, in that one does not obtain new Milnor fibers from non-normal representatives of their topological type (see Remark 6.1).

Acknowledgements. I benefited from conversations with Burak Ozbagci, Jan Stevens and Jean-Yves Welschinger. I am grateful also to the referee for his remarks and to János Kollár who, after seeing the first version of this paper put on ArXiv, communicated me a list of papers dealing at least partially with smoothings of non-normal isolated surface singularities (see Remark 5.10). This research was partially supported by the grant ANR-12-JS01-0002-01 SUSI and by Labex CEMPI (ANR-11-LABX-0007-01).

2. Generalities on normal surface singularities

In this section we recall the basic properties and classes of normal surface singularities which are needed in the sequel. More detailed introductions to the study of normal surface singularities are contained in [41], [28], [29], [51], [38].

Recall first the basic definition, valid in arbitrary dimension:

Definition 2.1. Let (X, x) be a germ of reduced complex analytic space. It is called **normal** if and only if its local ring of holomorphic functions is integrally closed in its total ring of fractions.

Normality may be characterized also in the following ways (see [52, Page 81]):

Proposition 2.2. *Let (X, x) be a germ of reduced complex analytic space. The following statements are equivalent:*

(1) *(X, x) is normal.*
(2) *The singular locus $S(X)$ of X is of codimension at least 2 and any holomorphic function on $X \setminus S(X)$ extends to a holomorphic function on X.*
(3) *Every bounded holomorphic function on $X \setminus S(X)$ extends to a holomorphic function on X.*

Using this proposition, it is easy to show that:

Corollary 2.3. *If the reduced germ (X, x) is normal, then every continuous function*

$$f : (X, x) \to (Y, y),$$

where (Y, y) is another holomorphic germ, is necessarily holomorphic whenever it is holomorphic on the complement of a nowhere dense closed analytic subspace $(X', x) \subset (X, x)$.

Any reduced germ has a canonical *normalization*, whose multilocal ring (direct sum of a finite collection of local rings) is the integral closure of the initial local ring in its total ring of fractions. It may be characterized in the following way:

Proposition 2.4. *Let (X, x) be a germ of reduced complex analytic space. There exists, up to unique isomorphism above (X, x), a unique finite morphism $\nu : (\tilde{X}, \tilde{x}) \to (X, x)$ from a finite disjoint union of germs to (X, x) (here $\tilde{x}$ denotes a finite set of points), such that:*

- *ν is an isomorphism outside the non-normal locus of X.*
- *$\tilde{X}$ is normal.*

Therefore, normal germs are necessarily irreducible. The normalization separates the irreducible components and eliminates the components of their singular loci which are of codimension 1. In particular, normal curve singularities are precisely the smooth ones and normal surface singularities are necessarily isolated. The converse is not true in any dimension (see the explanations given in the proof of Proposition 4.3). Nevertheless, complete intersection isolated singularities of dimension 2 or higher are necessarily normal (being Cohen-Macaulay, see the same proof). This is the reason why it is more difficult to exhibit examples of isolated non-normal singularities in dimension 2 or higher than in dimension 1.

Normal singularities are of fundamental importance even if one is interested in non-normal ones: a way to study them is through their morphism ν of *normalization*, characterized in the previous proposition. For much more details about normal varieties and the normalization maps, one may consult Greco's book [9].

One has a preferred family of representatives of any germ with isolated singularity:

Definition 2.5. Let (X, x) be a germ of reduced and irreducible complex analytic space with isolated singularity. Choose a representative of it embedded in $(\mathbb{C}^n, 0)$. Consider the euclidean sphere $\mathbb{S}^{2n-1}(r) \subset \mathbb{C}^n$ of radius $r > 0$, centered at 0. Denote by $\mathbb{B}^{2n}(r)$ the ball bounded by it. A ball $\mathbb{B}^{2n}(r_0)$ is called a **Milnor ball** if all the spheres of radius $r \in (0, r_0]$ are transversal to the representative. In this case, the intersection $X \cap \mathbb{B}^{2n}(r_0)$ is called a **Milnor representative** of the germ and $X \cap \mathbb{S}^{2n-1}(r_0)$ is the **boundary** of the germ.

The boundary is independent, up to diffeomorphisms preserving the orientation, of the choices done in this construction (see Looijenga [24]). We will denote its oriented diffeomorphism type, or a representative of it, by $\partial(X, x)$. One may show, moreover, that the boundary of an isolated singularity is isomorphic to the boundary of its normalization. This may seem obvious intuitively, as the normalization morphism is in this case an isomorphism outside the singular point, but one has to work more, because the lift to the normalization of the euclidean distance function serving to define the intersections with spheres for the initial germ are not euclidean distance functions for the normalization. For a detailed treatment of this issue, see [6].

Let us fix a Milnor ball $\mathbb{B}^{2n}(r_0)$. At each point of the representative $X \cap \mathbb{S}^{2n-1}(r_0)$ of $\partial(X, x)$, consider the maximal subspace of the tangent space which is invariant by the complex multiplication. It is a (real) hyperplane, canonically oriented by the complex multiplication. This field of hyperplanes is moreover a *contact structure*, as a consequence of the fact that the spheres by which we intersect are strongly pseudoconvex. In fact, this oriented contact manifold is also independent of the choices. We call it the **contact boundary** $(\partial(X, x), \xi(X, x))$ of the singularity (X, x) (for details, see [6]). In the same reference, we introduced the following terminology:

Definition 2.6. An oriented (contact) manifold is called **Milnor fillable** if it is isomorphic to the (contact) boundary of an isolated singularity.

From now on, we will restrict to surfaces. One of the most important tools to study them is:

Definition 2.7. Let $(S, 0)$ be a normal surface singularity which is not smooth. A **resolution** of it is a morphism $\pi : (\Sigma, E) \to (S, 0)$, where E denotes the preimage of 0 by π, such that:

- π is proper;
- Σ is smooth;
- π is an isomorphism from $\Sigma \setminus E$ to $S \setminus 0$.

The subset E of Σ, which is always a connected divisor, is called the **exceptional divisor** of Σ. If E is a divisor with normal crossings whose irreducible components are smooth, we say that π is a **simple normal crossings (snc)** resolution. In this last case, the **dual graph** of the resolution has as vertices the irreducible components of E, the edges being in bijection with the intersection points of those components.

Note that the hypothesis of having simple normal crossings prohibits the existence of loops in the dual graphs, but not that of multiple edges. In fact, the number of edges between two vertices is equal to the intersection number of the corresponding components.

There always exist resolutions. Moreover, there is always a *minimal* snc resolution, unique up to unique isomorphism above $(S, 0)$, the minimality meaning that any other snc resolution factors through it. It is this resolution which is most widely used for the topological study of the boundary of the singularity. Nevertheless, for its algebraic study, sometimes it is important to work with the *minimal resolution*, in which we don't ask any more the exceptional resolution to have normal crossings or smooth components (see an example in Theorem 2.15). It is again a theorem that such a resolution also exists up to unique isomorphism.

If π is a resolution of $(S, 0)$, denote by $\mathrm{Eff}(\pi)$ the free abelian semigroup generated by the irreducible components of its exceptional divisor, that is, the additive semigroup of the integral effective divisors supported by E. If $Z_1, Z_2 \in \mathrm{Eff}(\pi)$, we say that Z_1 **is less than** Z_2 if $Z_2 - Z_1$ is also effective and $Z_1 \neq Z_2$. We write then $Z_1 < Z_2$.

Proposition 2.8. *Let π be any resolution of the normal surface singularity $(S, 0)$. There exists a non-zero cycle $Z_{num} \in \mathrm{Eff}(\pi)$, called the* **numerical cycle** *of π, which intersects non-positively all the irreducible components of E, and which is less than all the other cycles having this property.*

By definition, the numerical cycle is unique, once the resolution is fixed. It was defined first by M. Artin [2], and Laufer [19] gave an algorithm to compute it.

We will need a second cycle supported by E, this time with *rational* coefficients, possibly non-integral.

Proposition 2.9. *Let $\pi : (\Sigma, E) \to (S, 0)$ be any resolution of the normal surface singularity $(S, 0)$. There exists a unique cycle Z_K supported by E, with rational coefficients, such that $Z_K \cdot E_i = -K_\Sigma \cdot E_i$ for any component E_i of E. It is called the* **anticanonical cycle** *of π. Here K_Σ denotes any canonical divisor of Σ.*

The canonical divisors on Σ are the divisors of the meromorphic 2-forms on a neighborhood of E in Σ. Such forms are precisely the lifts of the meromorphic 2-forms on a neighborhood of 0 in S. Of special importance are the normal surface singularities admitting such a 2-form which, moreover, is holomorphic and does not vanish on $S \setminus 0$:

Definition 2.10. An isolated surface singularity $(S, 0)$ is **Gorenstein** if it is normal and if it admits a non-vanishing holomorphic form of degree 2 on $S \setminus 0$.

In fact, isolated complete intersection surface singularities are not only normal, but also Gorenstein. We remark that the topological types of Gorenstein isolated surface singularities are known by [40], but it is an open question to describe the topological types of those which are complete intersections or hypersurfaces.

Both the anticanonical cycle and the notion of Gorenstein singularity are defined using differential forms of degree 2. Such forms are also useful to define several important notions of *genus*:

Definition 2.11. Let $(S, 0)$ be a normal surface singularity. Its **geometric genus** $p_g(S, 0)$ is equal to the dimension of the space of holomorphic 2-forms on $S \setminus 0$, modulo the subspace of forms which extend holomorphically to a resolution of S.

If Z is a compact divisor on a smooth complex surface Σ, its **arithmetic genus** $p_a(Z)$ is equal to $1 + \dfrac{1}{2} Z \cdot (Z + K_\Sigma)$.

In the same way as the *rational* curves are those of smooth algebraic curves of genus (in the usual Riemannian sense) 0, M. Artin [2] defined:

Definition 2.12. A normal surface singularity is **rational** if its geometric genus is 0.

By contrast with the case of curves, there is an infinite set of topological types of rational surface singularities. A basic property of them is that their minimal resolutions are snc, that all of the irreducible components of their exceptional divisors are rational curves, and that their dual graphs are trees. But this is not enough to characterize them. In fact, as proved by M. Artin [2]:

Proposition 2.13. *Let $(S, 0)$ be a normal surface singularity and let $\pi : (\Sigma, E) \to (S, 0)$ be any resolution of it. Then $(S, 0)$ is rational if and only if $p_a(Z_{num}) = 0$.*

The reader interested in the combinatorics of rational surface singularities may consult Lê and Tosun's paper [22] and Stevens' paper [48].

The singularities on which we focus in the sequel are not rational, as their resolutions contain non-rational exceptional curves:

Definition 2.14. A normal surface singularity is called **simple elliptic** if the exceptional divisor of its minimal resolution is an elliptic curve.

Simple elliptic singularities are necessarily Gorenstein, as a consequence of the following theorem of Laufer [20, Theorems 3.4 and 3.10]:

Theorem 2.15. *Let $(S, 0)$ be a normal surface singularity. Working with its minimal resolution, the following facts are equivalent:*

1. *One has $p_a(Z_{num}) = 1$ and $p_a(D) < 1$ for all $0 < D < Z_{num}$.*
2. *The fundamental and anticanonical cycles are equal: $Z_{num} = Z_K$.*
3. *One has $p_a(Z_{num}) = 1$ and any connected proper subdivisor of E contracts to a rational singularity.*
4. *$p_g(S, 0) = 1$ and $(S, 0)$ is Gorenstein.*

Laufer introduced a special name (making reference to condition (3)) for the singularities satisfying one of the previous conditions:

Definition 2.16. A normal surface singularity satisfying one of the equivalent conditions stated in Theorem 2.15 is called a **minimally elliptic** singularity.

In fact, as may be rather easily proved using characterization (3) of minimally elliptic singularities, the simple elliptic singularities are precisely the minimally elliptic ones which admit resolutions whose exceptional divisors have at least one non-rational component.

3. Generalities on deformations and smoothings of isolated singularities

In this section we recall the basic definitions and properties about deformations of isolated singularities which are needed in the sequel. For more details, one may consult Looijenga [24], Looijenga & Wahl [25], Stevens [46], Greuel, Lossen & Shustin [10] and Némethi [30].

Definition 3.1. Let (X, x) be a germ of a complex analytic space. A **deformation** of (X, x) is a germ of *flat* morphism $\psi : (Y, y) \to (S, s)$ together with an isomorphism between (X, x) and the special fiber $\psi^{-1}(s)$. The germ (S, s) is called the **base** of the deformation.

For example, when X is reduced, $f \in m_{X,x}$ is flat as a morphism $(X, x) \xrightarrow{f} (\mathbb{C}, 0)$ if and only if f does not divide zero, that is, if and only if f does not vanish on a whole irreducible component of (X, x). Such deformations over germs of smooth curves are called 1-*parameter deformations*. The simplest example is obtained when $X = \mathbb{C}^n$. Then one gets the prototypical situation considered by Milnor [26].

In general, to think about a flat morphism as a "deformation" means to see it as a family of continuously varying fibers (in the sense that their dimension is locally constant, without blowing-up phenomena) and to concentrate on a particular fiber, the nearby ones being seen as "deformations" of it. From such a family, one gets new families by rearranging the fibers, that is, by *base change*. One is particularly interested in the situations where there exist families which generate all other families by such base changes. The following definition is a reformulation of [10, Definition 1.8, page 234]:

Definition 3.2. (1) A deformation of (X, x) is **complete** if any other deformation is obtainable from it by a base-change.

(2) A complete deformation ψ of (X, x) is called **versal** if for any other deformation over a base (T, t) and identification of the induced deformation over a subgerm $(T', t) \hookrightarrow (T, t)$ with a pull-back from ψ, one may extend this identification with a pull-back from ψ over all (T, t).
(3) A versal deformation is **miniversal** if the Zariski tangent space of its base (S, s) has the smallest possible dimension.

When the miniversal deformation exists, its base space is unique *up to non-unique isomorphism* (only the tangent map to the isomorphism is unique). For this reason, one does not speak about a *universal* deformation, and was coined the word "miniversal", with the variant "semi-universal".

In many references, versal deformations are defined as the complete ones in the previous definition. Then is stated the theorem that the base of a versal deformation is isomorphic to the product of the base of a miniversal deformation and a *smooth* germ. But with this weaker definition the result is false. Indeed, starting from a complete deformation, by doing the product of its base with *any* germ (not necessarily smooth) and by taking the pull-back, we would get again a complete deformation. This shows that a complete deformation is not necessarily versal. Nevertheless, the theorem stated before is true if one uses the previous definition of *versality*.

Not all germs admit versal deformations. But those with isolated singularity do admit, as was proved by Schlessinger [43] for formal deformations (that is, over spectra of formal analytic algebras), then by Grauert [8] for holomorphic ones (an important point of this theorem being that one has to work with general analytic spaces, possibly non-reduced):

Theorem 3.3. *Let (X, x) be an isolated singularity. Then the miniversal deformation exists and is unique up to (non-unique) isomorphism.*

One may extend the notion of deformation by allowing bases of infinite dimension. Then even the germs with non-isolated singularity have versal deformations (see Hauser's papers [14], [15]).

In the sequel we will be interested in deformations with smooth generic fibers:

Definition 3.4. A **smoothing** of an isolated singularity (X, x) is a 1-parameter deformation whose generic fibers are smooth. A **smoothing component** of (X, x) is an irreducible component of the reduced miniversal base space over which the generic fibers are smooth.

Isolated complete intersection singularities have a miniversal deformation $(Y, y) \xrightarrow{\psi} (S, s)$ such that both Y and S are smooth, therefore irreducible (see [24]). In general, the *reduced miniversal base* (S_{red}, s) may be reducible. The first example of this phenomenon was discovered by Pinkham [37, Chapter 8]:

Proposition 3.5. *The germ at the origin of the cone over the rational normal curve of degree 4 in $\mathbb{P}^4$ has a reduced miniversal base space with two components, both being smoothing ones.*

Not all isolated singularities are smoothable. The most extreme case is attained with *rigid* singularities, which are not deformable at all in a non-trivial way. For example, quotient singularities of dimension ≥ 3 are rigid (Schlessinger [44]).

In [39] we proved a purely topological obstruction to smoothability for singularities of dimension ≥ 3. In dimension 2 no such criterion is known for *all* normal singularities. But there exist such obstructions for *Gorenstein* normal surface singularities as a consequence of the following theorem of Steenbrink [45]:

Theorem 3.6. *Let (X, x) be a Gorenstein normal surface singularity. If it is smoothable, then:*

$$\mu_- = 10\ p_g(X, x) - b_1(\partial(X, x)) + (Z_K^2 + |I|). \tag{3.1}$$

In the preceding formula, μ_- denotes the negative part of the index of the intersection form on the second homology group of *any* Milnor fiber (see Theorem 3.8 below) and $b_1(\partial(X,x))$ denotes the first Betti number of the boundary of (X,x). It may be computed from any snc resolution with exceptional divisor $E = \sum_{i\in I} E_i$ as:

$$b_1(\partial(X,x)) = b_1(\Gamma) + 2\sum_{i\in I} p_i,$$

where p_i denotes the genus of E_i and Γ denotes the dual graph of E. The term $Z_K^2 + |I|$ may also be computed using any snc resolution, and is again a topological invariant of the singularity.

The previous theorem implies that the expression in the right-hand side of (3.1) is ≥ 0, which gives non-trivial obstructions on the topology of smoothable normal Gorenstein singularities. For example, it shows that:

Proposition 3.7. *Among simple elliptic singularities, the smoothable ones have minimal resolutions whose exceptional divisor is an elliptic curve with self-intersection* $\in \{-9, -8, ..., -1\}$.

Proposition 3.7 has been proved first in another way by Pinkham [37, Chapter 7].

Let us look now at the topology of the generic fibers above a smoothing component. We want to localize the study of the family in the same way as Milnor localized the study of a function on $\mathbb{C}^n$ near a singular point. This is possible (see Looijenga [24]):

Theorem 3.8. *Let (X,x) be an isolated singularity. Let $(Y,y) \xrightarrow{\psi} (S,s)$ be a miniversal deformation of it. There exist (Milnor) representatives Y_{red} and S_{red} of the reduced total and base spaces of ψ such that the restriction $\psi : \partial Y_{red} \cap \psi^{-1}(S_{red}) \to S_{red}$ is a trivial C^∞-fibration. Moreover, one may choose those representatives such that over each smoothing component S_i, one gets a locally trivial C^∞-fibration $\psi : Y_{red} \cap \psi^{-1}(S_i) \to S_i$ outside a proper analytic subset.*

Hence, for each smoothing component S_i, the oriented diffeomorphism type of the oriented manifold with boundary $(\pi^{-1}(s) \cap Y_{red}, \pi^{-1}(s) \cap \partial Y_{red})$ does not depend on the choice of the generic element $s \in S_i$: it is called the **Milnor fiber** of that component. Moreover, its boundary is canonically identified with the boundary of (X,x) *up to isotopy*. In particular, the Milnor fiber of a smoothing component is diffeomorphic to a Stein filling of the contact boundary $(\partial(X,x), \xi(X,x))$.

Greuel and Steenbrink [11] proved the following topological restriction on the Milnor fibers of *normal* isolated singularities (of any dimension):

Theorem 3.9. *Let (X,x) be a normal isolated singularity. Then all its Milnor fibers have vanishing first Betti number.*

This is not true for non-normal isolated surface singularities, as may be seen for instance from the examples we give in the last section (see Remark 5.7).

For singularities which are not complete intersections, it is in general difficult even to construct non-trivial deformations or to decide if there exist smoothings. There is nevertheless a general technique of construction of smoothings, applicable to *germs of affine cones at their vertices*. Next section is dedicated to it.

4. Sweeping out the cone with hyperplane sections

In this section we recall Pinkham's method of construction of smoothings by "*sweeping out the cone with hyperplane sections*". It may be applied to the germs of affine cones at their vertices. The reader may follow the explanations on Figure 1.

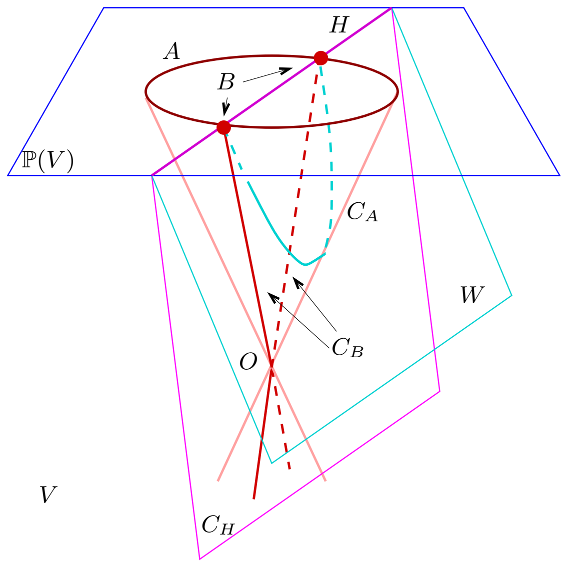

FIGURE 1. Sweeping the cone with hyperplane sections

Let V be a complex vector space, whose projectivisation is denoted $\mathbb{P}(V)$: set-theoretically, it consists of the lines of V. More generally, we define the *projectivisation* $\mathbb{P}(\mathcal{V})$ of a vector bundle $\mathcal{V}$ as the set of lines contained in the various fibers of the bundle. This notion will be used in the next section (see Remark 5.4).

Let A be a smooth subvariety of $\mathbb{P}(V)$. Denote by $C_A \hookrightarrow V$ the *affine cone* over it, and by $\overline{C_A} \hookrightarrow \overline{V}$ the associated *projective cone*. Here $\overline{V}$ denotes the projective space of the same dimension as V, obtained by adjoining $\mathbb{P}(V)$ to V as hyperplane at infinity. That is:

$$\overline{V} = \mathbb{P}(V \oplus \mathbb{C}) = V \cup \mathbb{P}(V).$$

The projective cone $\overline{C_A} = C_A \cup A$ is the Zariski closure of C_A in $\overline{V}$. The *vertex* of either cone is the origin O of V.

Assume now that $H \hookrightarrow \mathbb{P}(V)$ is a projective hyperplane which intersects A *transversally*. Denote by:

$$B := H \cap A$$

the corresponding hyperplane section of A. The affine cone C_H over H is the linear hyperplane of V whose projectivisation is H. The associated projective cone $\overline{C_H} \hookrightarrow \overline{V}$ is a projective hyperplane of $\overline{V}$.

Let L be the pencil of hyperplanes of $\overline{V}$ generated by $\mathbb{P}(V)$ and $\overline{C_H}$. That is, it is the pencil of hyperplanes of $\overline{V}$ passing through the "axis" H. In restriction to V, it consists in the levels of any linear form $f : V \to \mathbb{C}$ whose kernel is C_H. The 0-locus of $f|_{C_A}$ is the affine cone C_B over B.

As an immediate consequence of the fact that H intersects A transversally, we see that C_B has an isolated singularity at 0 and that all the non-zero levels of $f|_{C_A}$ are smooth. This shows that:

Lemma 4.1. *The map $f|_{C_A} : C_A \to \mathbb{C}$ gives a smoothing of the isolated singularity (C_B, O).*

Such are the smoothings obtained by "*sweeping out the cone with hyperplane sections*", in the words of Pinkham [37, Page 46]. It is probably the easiest way to construct smoothings, which explains why a drawing similar to the one we include here was represented on the cover of Stevens' book [46].

Since the complement $C_A \setminus O$ of the vertex in the cone C_A is homogeneous under the natural $\mathbb{C}^*$-action by scalar multiplication on V, the Milnor fibers of $f|_{C_A} : (C_A, O) \to (\mathbb{C}, 0)$ are diffeomorphic to the global (affine) fibers of $f|_{C_A} : C_A \to \mathbb{C}$. Those fibers are the complements $(W \cap \overline{C_A}) \setminus B$, for the members W of the pencil L different from $\overline{C_H}$ and $\mathbb{P}(V)$. But the only member of this pencil which intersects $\overline{C_A}$ non-transversally is $\overline{C_H}$, which shows that the pair $(W \cap \overline{C_A}, B)$ is diffeomorphic to $(\mathbb{P}(V) \cap \overline{C_A}, B) = (A, B)$. Therefore:

Proposition 4.2. *The Milnor fibers of the smoothing $f|_{C_A} : (C_A, O) \to (\mathbb{C}, 0)$ of the singularity (C_B, O) are diffeomorphic to the affine subvariety $A \setminus B$ of the affine space $\mathbb{P}(V) \setminus H$.*

The previous method may be applied to construct smoothings of germs of affine cones C_B at their vertices. In order to apply it, one has therefore to find another subvariety A of the same projective space, containing B, and such that B is a section of A by a hyperplane intersecting it transversally. In general, this is a difficult problem.

The important point to be understood here is that, *even if (C_A, O) is normal, this is not necessarily the case for its hyperplane section (C_B, O)*. More generally, if (Y, y) is a *normal* isolated singularity and $f : (Y, y) \to (\mathbb{C}, 0)$ is a holomorphic function such that the germ $(f^{-1}(0), y)$ is reduced and with isolated singularity, it is not necessarily normal. In dimension 3, in which we are especially interested in here, something special happens:

Proposition 4.3. *Assume that (Y, y) is a normal germ of 3-fold, with isolated singularity, and that $(f^{-1}(0), y)$ has also an isolated singularity. Then $(f^{-1}(0), y)$ is normal if and only if (Y, y) is Cohen-Macaulay.*

Proof. Let us explain first basic intuitions about *Cohen-Macaulay germs*. This notion appears naturally if one studies singularities using successive hyperplane sections. Intrinsically speaking, a hyperplane section of a germ (Y, y) is defined as the zero-locus of a function $f \in \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of the local ring $\mathcal{O}$ of the germ, endowed with the analytic structure given by the quotient local ring $\mathcal{O}/(f)$. This section is of dimension at least $\dim(Y, y) - 1$. Dimension drops necessarily if f *is not a divisor of* 0 in $\mathcal{O}$. Do such functions exist? Not necessarily. But if they exist, we take the hyperplane section and we repeat the process. (Y, y) is called *Cohen-Macaulay* if it is possible to drop in this way iteratively the dimension till arriving at an analytical space of dimension 0 (that is, set-theoretically, at the point y).

For the basic properties of the previous notion, one may consult [52] or [9]. Here we will need only the following facts:

(1) If a germ is Cohen-Macaulay, then for any $f \in \mathfrak{m}$ non-dividing 0, the associated hyperplane section $(f^{-1}(0), y)$ is also Cohen-Macaulay.
(2) An isolated surface singularity is normal if and only if it is Cohen-Macaulay.

Assume now that (Y, y) satisfies the hypothesis of the proposition.

- If (Y, y) is Cohen-Macaulay and if the hyperplane section $(f^{-1}(0), y)$ has an isolated singularity, property (1) implies that $(f^{-1}(0), y)$ is also Cohen-Macaulay. Property (2) implies then that it is normal.
- Conversely, if $(f^{-1}(0), y)$ is normal, then it is Cohen-Macaulay by property (2), which implies by definition that (Y, y) is also Cohen-Macaulay.

□

Let us come back to the smooth projective varieties $B \subset A \subset \mathbb{P}(V)$. The cone C_B is therefore not necessarily normal, even if C_A is. But its normalization is easy to describe:

Proposition 4.4. *The normalization of C_B is the algebraic variety obtained by contracting the zero-section of the total space of the line bundle $\mathcal{O}(-1)|_B$, which is isomorphic to the conormal line bundle of B in A.*

Proof. The isomorphism of the two line bundles follows from the fact that B is the vanishing locus of a section of $\mathcal{O}(1)|_A$. Here, as is standard in algebraic geometry, $\mathcal{O}(-1)$ denotes the dual of the tautological line bundle on $\mathbb{P}(V)$. Its fiber above a point of $\mathbb{P}(V)$ is the associated line.

Denote by $\tilde{C}_B$ the space obtained by contracting the zero-section of $\mathcal{O}(-1)|_B$, and by $\tilde{O} \in \tilde{C}_B$ the image of the 0-section. By the definition of contractions, $\tilde{C}_B$ is normal (see [52]). As the fiber of $\mathcal{O}(-1)|_B$ over a point $b \in B \hookrightarrow \mathbb{P}(V)$ is the line of V whose projectivisation is b, we see that there is a morphism:

$$\nu : \tilde{C}_B \to C_B$$

which induces an isomorphism $\tilde{C}_B \setminus \tilde{O} \simeq C_B \setminus O$. As $\tilde{C}_B$ is normal, by Corollary 2.3 and Proposition 2.4 we see that ν is a normalization morphism. □

5. Isolated singularities with simple elliptic normalization

In this section we apply the method of sweeping out the cone with hyperplane sections in order to show that the total space of the minimal resolution of any non-smoothable simple elliptic surface singularity is diffeomorphic to the Milnor fiber of some non-normal isolated surface singularity with simple elliptic normalization. We recall first several known properties of ruled surfaces over elliptic curves, following Hartshorne's presentation done in [13, Chapter V.2]. We conclude with a generalization valid for any normal surface singularity, using results of Laufer and Bogomolov & de Oliveira.

In order to apply the method of the previous section to singularities with simple elliptic normalization, we want to find surfaces embedded in some projective space which admit a transversal hyperplane section which is an elliptic curve. Moreover, because of Propositions 4.4 and 3.7, we would like to get an elliptic curve whose self-intersection number in the surface is ≥ 10. As a consequence of the following theorem of Hartshorne [12], this forces us to take a *ruled surface*:

Theorem 5.1. *Let C be a smooth compact curve of genus g on a smooth compact complex algebraic surface S. If $S \setminus C$ is minimal (that is, it does not contain smooth rational curves of self-intersection (-1)) and $C^2 \geq 4g + 6$, then S is a ruled surface and C is a section of the ruling.*

Ruled surfaces are those swept by lines (smooth rational curves):

Definition 5.2. A **ruled surface** above a smooth projective curve C is a smooth projective surface X together with a surjective morphism $\pi : X \to C$, such that all (scheme-theoretic) fibers are isomorphic to $\mathbb{P}^1$.

It is a theorem that all ruled surfaces admit regular sections.

The following theorem is basic for the classification of ruled surfaces (see [13, Prop. V.2.8, V.2.9]):

Theorem 5.3. *If $\pi : X \to C$ is a ruled surface, it is possible to write $X \simeq \mathbb{P}(\mathcal{E}^*)$, where $\mathcal{E}$ is a plane bundle on C with the property that $H^0(\mathcal{E}) \neq 0$, but for all line bundles $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$, we have $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$. In this case the integer $e = -\deg \mathcal{E}$ is an invariant* *of X. Furthermore, in this case there is a section $\sigma_0 : C \to X$ with image C_0, such that $\mathcal{O}_X(C_0) \simeq \mathcal{O}_X(1)$. One has $C_0^2 = -e$.*

In the sequel, we will say that e is the *numerical invariant* of the ruled surface.

Remark 5.4. In fact, Hartshorne writes $\mathbb{P}(\mathcal{E})$ instead of $\mathbb{P}(\mathcal{E}^*)$. The reason is that his definition of projectivisation is dual to the one we use in this paper: instead of taking the lines in a vector space or vector bundle, he takes the hyperplanes, that is, the lines in the dual vector space/bundle.

We want to find sections of ruled surfaces which appear as hyperplane sections for some embedding in a projective space, that is, according to a standard denomination of algebraic geometry, *very ample* sections. The following proposition combines results contained in [13, Theorems 2.12, 2.15, Exercice 2.12 of Chapter V]:

Proposition 5.5. *Assume that C is an elliptic curve and that X is a ruled surface above C with numerical invariant e. Then:*

(1) *When X varies for fixed C, the invariant e takes all the values in $\mathbb{Z} \cap [-1, \infty)$.*
(2) *Consider a fixed such ruled surface and let F be one of its fibers. Take $a \in \mathbb{Z}$. Then the divisor $C_0 + aF$ is very ample on X if and only if $a \geq e + 3$.*

Fix now an integer $a \geq e+3$. By Proposition 5.5, the divisor $C_0 + aF$ is very ample. Denote by $X \hookrightarrow \mathbb{P}(V)$ the associated projective embedding. Let H be a hyperplane which intersects it transversally, and let $B := H \cap X$. Therefore B is linearly equivalent to $C_0 + aF$ on X. We have the following intersection numbers on X:

$$\begin{cases} B \cdot F = (C_0 + aF) \cdot F = C_0 \cdot F = 1 \\ B^2 = (C_0 + aF)^2 = C_0^2 + 2aC_0 \cdot F = -e + 2a. \end{cases}$$

We have used the facts that:

- F is a fiber, which implies that $F^2 = 0$;
- C_0 is a section, which implies that $C_0 \cdot F = 1$;
- $C_0^2 = -e$, by Theorem 5.3.

The first equality above implies that B is again a section of the ruled surface. The second equality shows that a tubular neighborhood of B in X is diffeomorphic to a disc bundle over C with Euler number $-e+2a$. As B is a section of the ruling, such a disc bundle may be chosen as a differentiable sub-bundle of the ruling. As the fibers of the ruling $\pi : X \to C$ are spheres, its complement is again a disc bundle, necessarily of opposite Euler number. Proposition 4.2 shows then that:

Proposition 5.6. *The Milnor fiber of the smoothing $f|_{C_A} : (C_X, 0) \to (\mathbb{C}, 0)$ of the isolated surface singularity $(C_B, 0)$ is diffeomorphic to the disc bundle over C with Euler number $e - 2a$.*

Remark 5.7. This shows that the first Betti number of the Milnor fiber of this smoothing is 2. Greuel and Steenbrink's theorem 3.9 implies that the surface singularity $(C_B, 0)$ which is being smoothed is non-normal.

By Proposition 5.5, we see that the integer $e - 2a$ takes any value in $\mathbb{Z} \cap (-\infty, -5]$ (because for fixed e, it takes all the integral values in $(-\infty, -e-6]$ which have the same parity as $-e-6$). Therefore:

- this construction applies to simple elliptic singularities whose minimal resolution has an exceptional divisor with self-intersection any number in $\mathbb{Z} \cap (-\infty, -5]$;
- the Milnor fiber is diffeomorphic to the minimal resolution, both being diffeomorphic to the disc bundle over C with Euler number $e - 2a$.

More generally, as an easy consequence of results of Laufer [21] and Bogomolov and de Oliveira [5], we have:

Proposition 5.8. *Let $(S, 0)$ be any normal surface singularity. Then there exists an isolated surface singularity with normalization isomorphic to $(S, 0)$, which has a smoothing whose Milnor fibers are diffeomorphic to the minimal resolution of $(S, 0)$.*

Proof. Choose a Milnor representative of $(S, 0)$ (see Definition 2.5). Therefore its boundary is strongly pseudo-convex. Take the minimal resolution $\pi : (\Sigma, E) \to (S, 0)$. As π is an isomorphism outside 0, the boundary of Σ is also strongly pseudo-convex. By the extensions done in [5] of Laufer's results of [21], there exists a 1-parameter deformation:

$$\psi : (\tilde{\Sigma}, \Sigma) \to (\mathbb{D}_\epsilon, 0)$$

of Σ over a disc $\mathbb{D}_\epsilon$ of radius $\epsilon > 0$, such that the fibers Σ_t of ψ above any point $t \in \mathbb{D}_\epsilon \setminus 0$ do not contain compact curves. If we choose the disc $\mathbb{D}_\epsilon$ small enough, the boundaries of those fibers are also strongly pseudoconvex, by the stability of this property. Therefore, the fibers of ψ above $\mathbb{D}_\epsilon \setminus 0$ are all Stein.

Consider now the *Remmert reduction* (see [52, Page 229]):

$$\rho : \tilde{\Sigma} \to \tilde{S}.$$

By definition, it contracts all the maximal connected compact analytic subspaces of $\tilde{\Sigma}$ to points, and it is normal. The only compact curve of $\tilde{\Sigma}$ is E, therefore ρ contracts E to a point P, $\tilde{S}$ is a normal 3-fold and ρ is an isomorphism above $\tilde{S} \setminus P$. As $\tilde{S}$ is normal, Corollary 2.3 shows that the map ψ descends to it, giving us a family:

$$\psi' : (\tilde{S}, S') \to (\mathbb{D}_\epsilon, 0).$$

Here S' denotes the fiber of ψ' above the origin. The map ρ being an isomorphism in restriction to $\tilde{\Sigma} \setminus E$, it gives an isomorphism:

$$\Sigma \setminus E \simeq S' \setminus P.$$

Composing it with the isomorphism $\pi^{-1} : S \setminus 0 \to \Sigma \setminus E$, we get an isomorphism $S \setminus 0 \simeq S' \setminus P$ which extends by continuity to S. As S is normal, we see that $(S, 0)$ is indeed the normalization of (S', P).

The map ψ' gives therefore a smoothing with the desired properties:

- Its central fiber (S', P) has normalization isomorphic to $(S, 0)$.
- Its Milnor fibers are diffeomorphic to the total space of the minimal resolution of $(S, 0)$. Indeed, by construction they are isomorphic to the fibers of ψ. But ψ is a deformations of a smooth surface, therefore, by Ehresmann's theorem, *all* its fibers are diffeomorphic, and the central fiber is the minimal resolution Σ of S.

□

Compared with the general result 5.8, the advantage of the construction explained before for simple elliptic singularities, using the method of sweeping a cone with hyperplane sections, is that it shows that in that case the minimal resolution is diffeomorphic to an affine algebraic surface.

Remark 5.9. Laufer proved that one can find a 1-parameter deformation of the total space of the minimal resolution which destroys *any* irreducible component of the exceptional divisor. As for simple elliptic singularities the exceptional divisor is irreducible, we could use his result and proceed as in the previous proof, in order to get the proposition for this special class of singularities.

Remark 5.10. After seeing the first version of this paper put on ArXiv, János Kollár communicated me some information I did not know about papers dealing, at least partially, with the smoothability of non-normal isolated singularities. One of the earliest papers he may think about concerned with this problem is [27, Section 4]. There, Mumford gives examples of such surface singularities with simple elliptic normalizations (without looking at their Milnor fibers). Extending a result of Mumford's paper, Kollár proved in [16, Lemma 14.2] that all smoothings of isolated surface singularities with rational normalization lift to smoothings of the normalization: more precisely, with this hypothesis, if their total space is normal, then the special fiber is rational. This shows that *for rational surface singularities, one cannot obtain new Milnor fibers by the method of the present paper*. In higher dimensions, Kollár proved in [16, Theorem 3 (2)] that, if X_0 is a non-normal isolated singularity of dimension at least 3 whose normalization is log canonical, then X_0 is not smoothable: it does not even have normal deformations. He also indicated [18, Section 3.1] as a reference for basic material about singularities of cones.

6. Open questions

The following questions are basic for the understanding of the topology of the Milnor fibers of isolated, not necessarily normal surface singularities:

(1) Given a Milnor fillable contact 3-manifold, determine whether, up to diffeomorphisms/ homeomorphisms relative to the boundary, there is always a finite number of Milnor fibers corresponding to smoothings of not-necessarily normal isolated surface singularities filling it.
(2) Given a Milnor fillable contact 3-manifold (M, ξ), determine whether there exists an isolated surface singularity which fills it, such that its Milnor fibers exhaust, up to diffeomorphisms/ homeomorphisms, the Milnor fibers of the various isolated singularities which fill (M, ξ).
(3) Given a Milnor fillable contact 3-manifold (M, ξ), determine whether there exists an isolated surface singularity which fills it, such that its Milnor fibers exhaust, up to diffeomorphisms/homeomorphisms, the Stein fillings of (M, ξ).
(4) Given a Milnor fillable contact 3-manifold (M, ξ), classify, up to diffeomorphisms/homeomorphisms relative to the boundary, the Milnor fibers of the isolated singularities filling it, and determine the subset of those which appear as Milnor fibers of *normal* singularities.
(5) Determine bounds on the first Betti number of the Milnor fibers of an isolated non-normal surface singularity in terms of its analytic invariants.

Remark 6.1. For cyclic quotient singularities, Lisca [23] proved that there is a finite number of Stein fillings of their contact boundaries and he classified them up to diffeomorphisms relative to the boundary. He conjectured that they are diffeomorphic to the Milnor fibers of the corresponding singularity. Némethi and the present author proved this conjecture in [31]. Therefore, in this case the answers of the first three questions are positive and the fourth question is also answered. It would be interesting to understand if the fact that the first three questions have a positive answer is rather an exception or the rule for rational surface singularities. In this case, one does not need to look at non-normal representatives of the topological types, by Kollár's result [16, Lemma 14.2] cited in Remark 5.10.

References

[1] Akhmedov, A., Ozbagci, B. *Exotic Stein fillings with arbitrary fundamental groups.* arXiv: 1212.1743v1
[2] Artin, M. *On isolated rational singularities of surfaces.* Amer. J. Math. **88** (1966), 129-136. DOI: 10.2307/2373050

[3] Bhupal, M., Ono, K. *Symplectic fillings of links of quotient surface singularities.* Nagoya Math. J. **207** (2012), 1-45.

[4] Bhupal, M., Stipsicz, A. *Smoothings of singularities and symplectic topology.* In *Deformations of surface singularities.* A. Némethi and A. Szilard eds., Bolyai Soc. Math. Studies **23**, Springer, 2013, 57-97.

[5] Bogomolov, F.A., de Oliveira, B. *Stein Small Deformations of Strictly Pseudoconvex Surfaces.* Contemporary Mathematics **207** (1997), 25-41. DOI: 10.1090/conm/207/02717

[6] Caubel, C., Némethi, A., Popescu-Pampu, P. *Milnor open books and Milnor fillable contact 3-manifolds.* Topology **45** (2006), 673-689. DOI: 10.1016/j.top.2006.01.002

[7] Fintushel, R., Stern, R. J. *Rational blowdowns of smooth* 4*-manifolds.* J. Differential Geom. **46** (1997), no. 2, 181-235.

[8] Grauert, H. *Über die Deformationen isolierter Singularitäten analytische Mengen.* Invent. Math. **15** (1972), 171-198. DOI: 10.1007/BF01404124

[9] Greco, S. *Normal varieties.* Notes written with the collaboration of A. Di Sante. Institutiones Mathematicae IV. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1978.

[10] Greuel, G.-M., Lossen, C., Shustin, E. *Introduction to singularities and deformations.* Springer, 2007.

[11] Greuel, G.-M., Steenbrink, J. *On the topology of smoothable singularities.* Proc. of Symp. in Pure Maths. **40** (1983), Part 1, 535-545.

[12] Hartshorne, R. *Curves with high self-intersection on algebraic surfaces.* Publ. Math. I.H.E.S. **36** (1969), 111-125. DOI: 10.1007/BF02684600

[13] Hartshorne, R. *Algebraic Geometry.* Springer, 1977. DOI: 10.1007/978-1-4757-3849-0

[14] Hauser, H. *An algorithm of construction of the semiuniversal deformation of an isolated singularity.* In *Singularities*, Part 1 (Arcata, Calif., 1981), 567-573, Proc. Sympos. Pure Math. **40**, Amer. Math. Soc., Providence, R.I., 1983.

[15] Hauser, H. *La construction de la déformation semi-universelle d'un germe de variété analytique complexe.* Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 1, 1-56.

[16] Kollár, J. *Flatness criteria.* J. Algebra **175** (1995), no. 2, 715-727.

[17] Kollár, J. *Grothendieck-Lefschetz type theorems for the local Picard group.* J. Ramanujan Math. Soc. **28A** (2013), 267-285.

[18] Kollár, J. *Singularities of the minimal model program.* With a collaboration of Sándor Kovács. Cambridge Tracts in Mathematics, **200**. Cambridge University Press, Cambridge, 2013.

[19] Laufer, H. *On rational singularities.* Amer. J. Math. **94** (1972), 597-608. DOI: 10.2307/2374639

[20] Laufer, H.B. *On minimally elliptic singularities.* Amer. J. of Math. **99** (6), 1257-1295 (1977). DOI: 10.2307/2374025

[21] Laufer, H.B. *Lifting cycles to deformations of two-dimensional pseudoconvex manifolds.* Trans. Amer. Math. Soc. **266** No. 1 (1981), 183-202. DOI: 10.1090/S0002-9947-1981-0613791-4

[22] Lê, D. T., Tosun, M. *Combinatorics of rational singularities.* Comment. Math. Helv. **79** (2004), no. 3, 582-604.

[23] Lisca, P. *On symplectic fillings of lens spaces.* Trans. Amer. Math. Soc. **360** (2008), no. 2, 765-799.

[24] Looijenga, E.N. *Isolated singular points on complete intersections.* London Math. Soc. Lecture Notes Series **77**, Cambridge Univ. Press, 1984. DOI: 10.1017/CBO9780511662720

[25] Looijenga, E.N., Wahl, J. *Quadratic functions and smoothing surface singularities.* Topology **25** (1986), no. 3, 261-291.

[26] Milnor, J. *Singular Points of Complex Hypersurfaces.* Princeton Univ. Press, 1968.

[27] Mumford, D. *Some footnotes to the work of C. P. Ramanujam.* In *C. P. Ramanujam–a tribute*, 247-262, Tata Inst. Fund. Res. Studies in Math. **8**, Springer, Berlin-New York, 1978.

[28] Némethi, A. *Five lectures on normal surface singularities.* Lectures delivered at the Summer School "*Low dimensional topology*", Budapest, Hungary, 1998. In *Low Dimensional Topology*, Bolyai Society Mathematical Studies **8**, 269-351, 1999.

[29] Némethi, A. *Invariants of normal surface singularities.* Contemporary Mathematics **354**, 161-208 AMS, 2004.

[30] Némethi, A. *Some meeting points of singularity theory and low dimensional topology.* In *Deformations of surface singularities.* A. Némethi and A. Szilard eds., Bolyai Soc. Math. Studies **23**, Springer, 2013, 109-162.

[31] Némethi, A., Popescu-Pampu, P. *On the Milnor fibers of cyclic quotient singularities.* Proc. London Math. Society (3) **101** (2010), 554-588. DOI: 10.1112/plms/pdq007

[32] Ohta, H., Ono, K. *Symplectic fillings of the link of simple elliptic singularities.* J. Reine Angew. Math. **565** (2003), 183-205.

[33] Ohta, H., Ono, K. *Simple singularities and symplectic fillings.* J. Differential Geom. **69** (2005), 1-42.

[34] Ohta, H., Ono, K. *Examples of isolated surface singularities whose links have infinitely many symplectic fillings.* J. Fixed Point Theory Appl. **3** (2008), no. 1, 51-56.

[35] Ozbagci, B. *On the topology of fillings of contact* 3*-manifolds.* To appear in the Proceedings of the Conference "*Interactions between low dimensional topology and mapping class groups*" that was held in July 1-5, 2013 at the Max Planck Institute for Mathematics, Bonn.

[36] Park, H., Shin, D., Stipsicz, A. I. *Normal complex surface singularities with rational homology disk smoothings.* arXiv: 1311.1929

[37] Pinkham, H. C. *Deformations of algebraic varieties with* $\mathbf{G}_m$ *action.* Astérisque, No. **20**. Société Mathématique de France, 1974.

[38] Popescu-Pampu, P. *The geometry of continued fractions and the topology of surface singularities.* Advanced Stud. in Pure Maths **46** (2007), 119-195.

[39] Popescu-Pampu, P. *On the cohomology rings of holomorphically fillable manifolds.* In *Singularities II. Geometric and Topological Aspects.* J. P. Brasselet et al. eds. Contemporary Mathematics **475**, AMS, 2008, 169-188.

[40] Popescu-Pampu, P. *Numerically Gorenstein surface singularities are homeomorphic to Gorenstein ones.* Duke Math. Journal **159** No. 3 (2011), 539-559. DOI: 10.1215/00127094-1433412

[41] Reid, M. *Chapters on Algebraic Surfaces.* In *Complex Algebraic Geometry.* J. Kollár editor, AMS, 1997, 3-159.

[42] Saito, K. *Einfach-elliptische Singularitäten.* Invent. Math. **23** (1974), 289-325. DOI: 10.1007/BF01389749

[43] Schlessinger, M. *Functors of Artin rings.* Trans. Amer. Math. Soc. **130** (1968), 208-222. DOI: 10.1090/S0002-9947-1968-0217093-3

[44] Schlessinger, M. *Rigidity of quotient singularities.* Invent. Math. **14** (1971), 17-26. DOI: 10.1007/BF01418741

[45] Steenbrink, J. *Mixed Hodge structures associated with isolated singularities.* Proc. of Symposia in Pure Maths. **40** (1983), Part 2, 513-536.

[46] Stevens, J. *Deformations of singularities.* Springer LNM **1811**, 2003.

[47] Stevens, J. *Deforming nonnormal surface singularities.* Manuscript, 2009.

[48] Stevens, J. *On the classification of rational surface singularities.* J. Singul. **7** (2013), 108-133.

[49] Stipsicz, A. I., Szabó, Z., Wahl, J. *Rational blowdowns and smoothings of surface singularities.* J. Topol. **1** (2008), no. 2, 477-517.

[50] Varchenko, A.N. *Contact structures and isolated singularities.* Mosc. Univ. Math. Bull. **35** no.2 (1980), 18-22.

[51] Wall, C.T.C. *Quadratic forms and normal surface singularities.* In *Quadratic forms and their applications. (Dublin 1999)*, Contemp. Math. **272**, 293-311, AMS (2000).

[52] *Several complex variables VII.* Grauert, H., Peternell, Th., Remmert, R. eds. *Encyclopaedia of Mathematical Sciences* **74**, Springer, 1994.

Université Lille 1, UFR de Maths., Bâtiment M2, Cité Scientifique, 59655, Villeneuve d'Ascq Cedex, France.

E-mail address: patrick.popescu@math.univ-lille1.fr